A hyperdoctrinal reconstruction of conditional calculus

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Summary

- Bruggink et al. (2011) introduced conditional calculus for reasoning about reactive systems.
- It seems to be a thing in the graph rewriting community, relying on Nick for this.

Contribution

A high-level, equivalent definition in terms of hyperdoctrines.

Plan

Recap on hyperdoctrines

Heyting algebra: cartesian closed poset with finite colimits.

Notation

Heyt: Heyting algebras and functors with both left and right adjoints (in particular, bicontinuous).

Definition (Hyperdoctrine)

Functor

$$P\colon \mathbb{C}^{op} \to \mathbf{Heyt}$$

from some base category $\mathbb{C}$.

Intuition

- Object $A \in \mathbb{C}$: contexts.
- Morphism $f : A \to B$: substitutions.
- $P(A)$: propositions in context A.
- $P(f) : P(B) \to P(A)$: substitution/instantiation.
- Left adjoint $\exists_f : P(A) \to P(B)$: existential quantification.
- Right adjoint $\forall_f : P(A) \to P(B)$: universal quantification.

Notation

$f^* := P(f)$.

Example

Subset hyperdoctrine

$$\begin{aligned} \mathcal{P} \colon \mathbf{Set}^{op} &\to \mathbf{Set} \\ X &\mapsto \mathcal{P}(X) \\ (f \colon X \to Y) &\mapsto (f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)). \end{aligned}$$

- Left adjoint: image, i.e.,

$$\exists_f(\varphi) = f(\varphi) = \{y \mid \exists x, y = f(x) \wedge x \in \varphi\}.$$

- Right adjoint:

$$\forall_f(\varphi) = \{y \mid \forall x, y = f(x) \Longrightarrow x \in \varphi\}.$$

The subhom hyperdoctrine on a category $\mathbb{C}$

Definition

The subhom hyperdoctrine (on $\mathbb{C}^{op}$) is the composite

$$\mathbb{C} \xrightarrow{\coprod_{X\in\mathbb{C}} \mathbf{y}X} \mathbf{Set}^{op} \xrightarrow{\mathcal{P}} \mathbf{Set}.$$

Concretely:

$$\mathcal{P}_{\mathbb{C}}(C) = \mathcal{P}\left(\coprod_{X\in\mathbb{C}} \mathbb{C}(C,X)\right)$$

= sets of morphisms from C.

Why is this a hyperdoctrine?

Precomposition

Lemma

Hyperdoctrines are closed under precomposition by arbitrary functors.

Proof sketch.

Childish. □

Corollary

For any $\mathbb{C}$, $\mathcal{P}_{\mathbb{C}}$ is a hyperdoctrine.

$$\mathbb{C} \xrightarrow{\coprod_{X \in \mathbb{C}} \mathbf{y}X} \mathbf{Set}^{op} \xrightarrow{\mathcal{P}} \mathbf{Set}$$

Conditions on a fixed category $\mathbb{C}$

Definition (my notation)

Conditions φ are defined inductively by the following inference rule,

$$\frac{\dots \qquad f_i : A \to B_i \qquad B_i \vdash \varphi_i \qquad \dots \qquad (i \in I)}{A \vdash \varepsilon_{i \in I}(f_i, \varphi_i)}$$

where

- I denotes any set, and
- ε ranges over quantifiers, i.e., elements of $\{\forall, \exists\}$.

Let $\mathbf{Cond}_{\mathbb{C}}(A) =$ set of conditions φ over A, i.e., such that $A \vdash \varphi$.

Notation

We often omit the base objects of conditions, writing φ instead of $A \vdash \varphi$, when it is clear from context.

Semantics

Remark

The subhom hyperdoctrine has infinite conjunction and disjunction.

(Since $\mathcal{P}$ does.)

Definition

Satisfaction $[\![-]\!]_A : \mathbf{Cond}_{\mathbb{C}}(A) \to \mathcal{P}_{\mathbb{C}}(A)$ is defined inductively:

$$[\![\forall_{i\in I}(f_i, \varphi_i)]\!]_A = \bigwedge_{i\in I} \forall_{f_i} [\![\varphi_i]\!]_{B_i}$$

$$[\![\exists_{i\in I}(f_i, \varphi_i)]\!]_A = \bigvee_{i\in I} \exists_{f_i} [\![\varphi_i]\!]_{B_i}$$

(assuming $f_i : A \to B_i$).

Implicit base cases: $[\![\forall_{i\in\emptyset}\star]\!]_A = \top_A$ and $[\![\exists_{i\in\emptyset}\star]\!]_A = \bot_A$

Fundamental theorem of conditional calculus

Proposition

The image of conditions is closed under all hyperdoctrine operations, except perhaps instantiation, $\mathcal{P}_{\mathbb{C}}(f)\colon \mathcal{P}_{\mathbb{C}}(B) \to \mathcal{P}_{\mathbb{C}}(A)$, for $f\colon A \to B$.

Proof sketch

Easy, see Bruggink et al.

Theorem

If $\mathbb{C}$ has representative squares, then the image of condition is closed under instantiation.

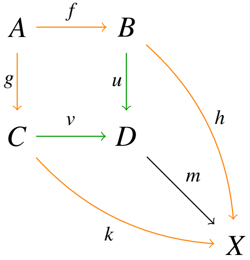
Proof sketch

Bruggink et al.'s shift operation, which relies on representative squares, coming up just next!

Representative squares

Definition

A class κ of squares in $\mathbb{C}$ is representative iff every outer square as below factors as shown with inner square in κ.



Example

The class of (resp. weak) pushouts, if they exist.

Proof sketch

Given a class κ of representative squares, conditions are closed under substitution.

Proof sketch

Bruggink et al. define substitution syntactically, and prove that it is

- left adjoint to $\forall$ and
- right adjoint to $\exists$.

Corollary

When $\mathbb{C}$ is equipped with a class of representative squares, Conditions induce a sub-hyperdoctrine of $\mathcal{P}_{\mathbb{C}}$.

Frobenius reciprocity: quantification vs conjunction and disjunction

Definition

A hyperdoctrine $P\colon \mathbb{C}^{op} \to \mathbf{Heyt}$ satisfies Frobenius reciprocity iff for all $f\colon A \to B$ and $\varphi, \psi \in P(B)$, the canonical morphism

$$f^*(\psi^{\varphi}) \to f^*(\varphi)^{f^*(\varphi)}$$

is an iso.

Proposition

Frobenius reciprocity is closed under precomposition.

Corollary

The subhom hyperdoctrine $\mathcal{P}_{\mathbb{C}}$ satisfies Frobenius reciprocity.

Beck-Chevalley: quantification vs instantiation

Definition

A commuting square is Beck-Chevalley iff its mate is an isomorphism.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow[v]{} & D \end{array} \qquad\qquad \begin{array}{ccc} PA & \xrightarrow{\exists_f} & PB \\ {\scriptstyle Pg}\uparrow & \leq & \uparrow{\scriptstyle Pu} \\ PC & \xrightarrow[\exists_v]{} & PD \end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \leq v^*\exists_v\varphi}\,\eta}{g^*\varphi \leq g^*v^*\exists_v\varphi}}{g^*\varphi \leq f^*u^*\exists_v\varphi}}{\exists_f g^*\varphi \leq u^*\exists_v\varphi}$$

Beck-Chevalley

Proposition

- **(BC$\mathcal{P}$)** Weak pullbacks are Beck-Chevalley in $\mathcal{P}$.
- **(BC$\coprod$)** Weak pullbacks are closed under coproducts in $\mathbf{Set}$.
- **(BC$\circ$)** Beck-Chevalley squares for a composite hyperdoctrine

$$\mathbb{C}^{op} \xrightarrow{F^{op}} \mathbb{D}^{op} \xrightarrow{P} \mathbf{Heyt}$$

are those mapped to Beck-Chevalley squares by F.

Beck-Chevalley

Corollary

If each $\mathbf{y}_X$ maps representative squares to weak pullbacks, then representative squares are Beck-Chevalley in $\mathcal{P}_{\mathbb{C}}$.

Example: (resp. weak) pushouts.

Proof

- By **(BC$\coprod$)**, $\coprod_X \mathbf{y}_X$ maps representative squares to weak pullbacks.
- Conclude by **(BC$\mathcal{P}$)** and **(BC$\circ$)**.

Conclusion

- Slightly abstract.
- Much easier technically than the original.
- Hope: useful for reasoning moves...

Nick, the floor is yours!