Algebraic and combinatorial aspects of category theory

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Part I

Invitation

Outline

1. Algebraic invitation

2. Combinatorial invitation

Outline

Part II

Correspondence between monads and theories, take 1

Outline

Outline

This lecture

A first, easy example bridge between algebraic and combinatorial category theory.

Equivalence between

- T-$\mathbf{Alg}$ for suitable monads T on sets and
- suitable presheaf categories.

A bit more detail

- Finitary monad, intuitively: finitary operations.
- Form a category $\mathbf{Mnd}_f(\mathbf{Set})$.
- Lawvere theory: small category with finite products freely generated by an object.
- Form a category $\mathbf{Law}$.
- Each Lawvere theory $\mathbb{L} \quad \mapsto \quad$ category $\mathbf{Mod}(\mathbb{L})$ of models.
- $\mathbf{Mod}(\mathbb{L}) \hookrightarrow [\mathbb{L}, \mathbf{Set}]$.

Theorem (Linton)

$$\begin{tikzcd}
\mathbf{Mnd}_f(\mathbf{Set}) \arrow[rr, "\simeq"] \arrow[dr, "(-)\text{-}\mathbf{Alg}"'] & & \mathbf{Law} \arrow[dl, "\mathbf{Mod}"] \\
& \mathbf{CAT} &
\end{tikzcd}$$

Reminder on monads

Definition

A monad on a category $\mathbf{C}$ is an endofunctor $T\colon \mathbf{C} \to \mathbf{C}$, equipped with natural transformations

$$\eta\colon id_{\mathbf{C}} \to T \qquad\qquad \mu\colon T \circ T \to T,$$

making the following diagrams commute.

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, equal] & T(T(X)) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, equal] \\
 & T(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\
T(T(X)) \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

Outline

Algebraic signatures

Definition (*Algebraic signature*)

Set O with map $a\colon O \to \mathbb{N}$.

Example

Let $\Sigma_{0,2}$ have

a constant e and a binary operation m,

given by

$$[0,2]\colon 2 \to \mathbb{N}.$$

Terms

Fix an algebraic signature Σ given by $a \colon O \to \mathbb{N}$.

Definition (Σ-*terms*)

- Family $\Sigma^* \colon \mathbf{Set} \to \mathbf{Set}$.
- Defined by $\Sigma^*(X) = \{X \vdash_\Sigma e\}$, where $\vdash_\Sigma$ defined inductively:

$$\dfrac{}{X \vdash_\Sigma [x]}\,\textsc{Var}\ (x \in X) \qquad \dfrac{X \vdash_\Sigma e_1 \quad \ldots \quad X \vdash_\Sigma e_p}{X \vdash_\Sigma o(e_1, \ldots, e_p)}\,\textsc{Op}\ (a(o) = p)$$

Syntactic categories

Definition (*Syntactic category* $\mathbb{L}_\Sigma$)

- Morphism $m \to n$:
  - n-tuple $\langle M_1, \ldots, M_n \rangle$,
  - where each $M_i \in \Sigma^*(m)$: term with scope $x_1, \ldots, x_m$ (names irrelevant).
- View each $M : m \to n$ as an *assignment* $[y_1 \mapsto M_1, \ldots, y_n \mapsto M_n]$.
- Composition $m \xrightarrow{M} n \xrightarrow{N} p$ by substitution

$$\langle N_1[M], \ldots, N_p[M] \rangle.$$

- Identity $n \to n$: $\langle x_1, \ldots, x_n \rangle$.

Syntactic categories

Exercise

Check the category axioms.

Syntactic categories

- Associativity of composition

$$m \xrightarrow{M} n \xrightarrow{N} p \xrightarrow{P} q$$

given by

$$P_l[N][M] = P_l[N[M]],$$

for all $l \in q$, where $N[M]_k = N_k[M]$ for all $k \in p$.

- Left unitality $id_n \circ M = M$:

$$(id_n \circ M)_j = y_j[M] = M_j \qquad (j \in n).$$

- Right unitality $M \circ id_m = M$:

$$(M \circ id_m)_j = M_j[\langle x_1, \ldots, x_m \rangle] = M_j \qquad (j \in n).$$

Syntactic categories

Exercise

Check the existence of

- binary products and
- terminal object

in $\mathbb{L}_\Sigma$.

Syntactic categories

Candidates:

- Product $m \times n$: sum $m + n$?!
- Terminal object: initial object 0?!

Syntactic categories

Proof.

- Morphisms $m \to 0$: 0-tuples of terms... i.e., $\langle\rangle$.
- Morphisms $m \to n + p$: $(n + p)$-tuples of terms with scope n.
  $\cong$ pairs of an n-tuple and a p-tuple.

$$\langle \underbrace{M_1, \ldots, M_n}_{n\text{-tuple}}, \underbrace{M_{n+1}, \ldots, M_{n+p}}_{p\text{-tuple}} \rangle$$

- Projections $n \xleftarrow{\langle x_1,\ldots,x_n \rangle} n + p \xrightarrow{\langle x_{n+1},\ldots,x_{n+p} \rangle} p$.

$$\langle x_1, \ldots, x_n \rangle [M] = \langle M_1, \ldots, M_n \rangle$$
$$\langle x_{n+1}, \ldots, x_{n+p} \rangle [M] = \langle M_{n+1}, \ldots, M_{n+p} \rangle.$$

□

Syntactic categories

Trivial observation:

Proposition

Every object in $\mathbb{L}_\Sigma$ is a *finite power*[a] of 1.

[a] Self, multiple product.

Proof.

Indeed, $n = 1 + \ldots + 1$, brilliant, I know.
Particular case: $0 = 1^0$. □

Towards Lawvere theories

Let us now abstract over the properties of syntactic categories.
$\rightsquigarrow$ Lawvere theories.

Outline

Finite products

Recall the definition of binary products?

Definition (*Product* of a family $(A_i)_{i \in I}$ of objects in category $\mathbf{C}$)

- Object $\prod_i A_i$ and projections $\pi_j \colon \prod_i A_i \to A_j$.
- Such that for all cones $\lambda_j : C \to A_j$,
- $\exists!$ cone morphism

$$\begin{tikzcd} C \arrow[rr, dashed] \arrow[dr, "\lambda_j"'] & & \prod_i A_i \arrow[dl, "\pi_j"] \\ & A_j . & \end{tikzcd}$$

A category has *finite products* if each $\prod_i A_i$ exists for finite I.

Finite products

Exercise

Show that $\mathbf{C}$ has finite products iff it has binary products and a terminal object.

Powers

Definition (I-th *power* A^I of $A \in \mathbf{C}$)

$\prod_{i \in I} A$.

Remark

Subtlety: power ≈ heterogeneous function object.
Indeed, I is a set!
Universal property:

$$\mathbf{C}(X, A^I) \cong \mathbf{Set}(I, \mathbf{C}(X, A)).$$

Skeletal categories

Definition (*Skeletal* category)

All isomorphic objects are equal.

Example

- Counterexample: $\mathbf{Set}_f$, the category of finite sets.
  Not small!
- Example: $\mathbb{F}$, finite cardinals (0, 1,...).

This is your first equivalence of categories!

Equivalence

Definition

An *equivalence of categories* is a pair

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "L"] & \mathbf{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

of functors, together with natural isomorphisms

$$\eta \colon id_{\mathbf{C}} \to RL \qquad \text{and} \qquad \varepsilon \colon LR \to id_{\mathbf{D}}.$$

Equivalence

Example

$$\begin{tikzcd} \mathbb{F} \arrow[r, hook, bend left] & \mathbf{Set}_f \arrow[l, bend left, "|-|"] \end{tikzcd}$$

- Choose bijection $\varepsilon \colon |X| \to X$ for any $X \in \mathbf{Set}_f$, identity on any $X \in \mathbb{F} \subseteq \mathbf{Set}_f$.
- $|-|$ on morphisms: $|f| := \varepsilon_Y^{-1} \circ f \circ \varepsilon_X$, so that we have:

$$\begin{tikzcd} {|X|} \arrow[r, "\varepsilon_X"] \arrow[d, dashed, "|f|"'] & X \arrow[d, "f"] \\ {|Y|} \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

- $\eta \colon n \to |n|$ is an equality (hence natural).

Interlude: full, faithful, embedding

For a functor $F : \mathbf{C} \to \mathbf{D}$, we have seen:

- full: $F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$ surjective $\forall A, B$;
- faithful: $F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$ injective $\forall A, B$.

Definition

Embedding: injective on objects + faithful.

$\approx$ subcategory.

Remark

Intuition for full embedding $E : \mathbf{C} \hookrightarrow \mathbf{D}$: $\mathbf{C} \cong$ image of E.

Characterisation of equivalences

Definition ($F \colon \mathbf{C} \to \mathbf{D}$ *essentially surjective on objects*)

Any $D \in \mathbf{D}$ is isomorphic to some $F(C)$.

Proposition

Any functor that is

- fully faithful and
- essentially surjective on objects

is an equivalence.

Exercise

- Prove this (requires the axiom of choice).
- Observe that the previous proof is an instance.

Lawvere theories

Definition (*Lawvere theory*)

- A small, skeletal category with finite products, whose objects all are finite powers of a single "generating" object.
- Morphism: functor preserving
  - finite products and
  - generating object.
- Form a category $\mathbf{Law}$.

Example

- Any $\mathbb{L}_{\Sigma}$.
- Particular case: empty Σ, say $\Sigma_{\emptyset}$.

Exercise: compute $\mathbb{L}_{\Sigma_{\emptyset}}$.

Solution

- Objects: finite cardinals, by definition.
- Morphisms $m \to n$: n-tuples of terms $m \vdash_\Sigma e$...
  but no operations $\rightsquigarrow$ terms $=$ variables.
- Thus, TFAE:
  - morphisms $m \to n$,
  - n-tuples of variables in $[i_1], \ldots, [i_n]$,
  - maps $n \to m$. (notational abuse: $m = \{1, \ldots, m\}$.)

We have shown:

Proposition

$\mathbb{L}_{\Sigma_\emptyset} \cong \mathbb{F}^{op}$, where $\mathbb{F} \hookrightarrow \mathbf{Set}$ denote the full subcategory on finite cardinals.

Pondering

Consequence

$\mathbb{F}^{op}$ is a Lawvere theory.

Get used to it:

A (small, skeletal) category with finite products whose objects all are finite powers of a single "generating" object.

- Disjoint sums yield finite coproducts in $\mathbb{F}$.
- All objects n are copowers $\coprod_{i \in n} 1$ of 1.
- Conclude by duality.

A syntactic viewpoint on $\mathbb{F}^{op}$

Think of morphisms $m \to n$ as renamings.

Observation

$f : n \to m$ in $\mathbb{F}$ vs $\langle \pi_{f(1)}, \ldots, \pi_{f(n)} \rangle : m \to n.$

(Each term is a variable, i.e., a projection from the scope.)

Example

$$
\begin{tikzcd}[row sep=small]
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & 2 \\
3 \arrow[uur, mapsto] & 3 \\
4 \arrow[ur, mapsto] &
\end{tikzcd}
$$

$$
\begin{array}{rcc}
4 & \to & 3 \\
\hline
1, \ldots, 3 & \mapsto & 1 \\
4 & \mapsto & 3
\end{array}
\qquad\qquad
\langle [1], [1], [1], [3] \rangle
$$

A syntactic viewpoint on $\mathbb{F}^{op}$

Proposition

$\mathbb{F}^{op}$ is initial in $\mathbf{Law}$.

Proof.

For any $\mathbb{L}$, any $\mathbb{F}^{op} \to \mathbb{L}$ needs to map

- 1 to generating object, say x,
- any $\langle \pi_{f(1)}, \ldots, \pi_{f(n)} \rangle \colon m \to n$ to

$$\langle \pi_{f(1)}, \ldots, \pi_{f(n)} \rangle \colon x^m \to x^n.$$

$\square$

Remark: $\mathbb{F}^{op} \to \mathbb{L}$ is bijective on objects.

Models of a Lawvere theory

Definition

Model of $\mathbb{L}$: finite product-preserving functor $\mathbb{L} \to \mathbf{Set}$.
Full subcategory $[\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}} \hookrightarrow [\mathbb{L}, \mathbf{Set}]$.

Remark

For model $M : \mathbb{L} \to \mathbf{Set}$,

$$M(n) = M(1^n) \cong M(1)^n.$$

Call $M(1)$ the carrier.

Forgetful functor $\mathbf{U}^{\mathbb{L}} : \mathbf{Mod}(\mathbb{L}) \to \mathbf{Set}$.

Moral

- For algebraic signature Σ, e.g., with $o\colon 2 \to 1$.
- $\mathbb{L}_\Sigma$ is the free category with finite products on Σ.
- So models $\mathbb{L}_\Sigma \to \mathbf{Set}$ correspond bijectively to choices of
  - a set X,
  - maps $X^n \to X$ for all operations of arity n.

Outline

Interlude I: adjunctions

- Fix monad $T \colon \mathbf{C} \to \mathbf{C}$.
- We have seen $\mathbf{U}^T \colon T\text{-}\mathbf{Alg} \to \mathbf{C}$.
- Intuition for $T(C)$: "free algebra".
- Let us substantiate and abstract.

Universal property of free algebra

Recall T-algebra structure $\mu^T_C \colon T(TC) \to TC$.

Proposition

For any T-algebra $a \colon TA \to A$, the map

$$T\text{-}\mathbf{Alg}(TC, A) \xrightarrow{\mathbf{U}^T_{TC,A}} \mathbf{C}(TC, A) \xrightarrow{\mathbf{C}(\eta^T_C, A)} \mathbf{C}(C, A)$$

is bijective.

$$\begin{tikzcd} C \arrow[r, "\eta^T_C"] \arrow[dr, "f"'] & TC \arrow[d, dashed, "\tilde{f}"] \\ & A \end{tikzcd}$$

Universal property of free algebra

$$\begin{tikzcd}
C \arrow[r, "\eta^T_C"] \arrow[dr, "f"'] & TC \arrow[d, dashed, "\tilde{f}"] \\
& A
\end{tikzcd}$$

Exercise

Prove this.

Candidate $\tilde{f}$

$$\begin{tikzcd}
C \arrow[r, "\eta^T_C"] \arrow[ddr, "f"'] & TC \arrow[d, "Tf"] \\
& TA \arrow[d, "a"] \\
& A
\end{tikzcd}$$

Need to check:

- commutation of triangle,
- algebra morphism,
- uniqueness.

Commutation of triangle

$$\begin{tikzcd}
C \arrow[rr, "\eta^T_C"] \arrow[dr, "f"'] & & TC \arrow[d, "Tf"] \\
& A \arrow[r, "\eta^T_A"] \arrow[dr, equal] & TA \arrow[d, "a"] \\
& & A
\end{tikzcd}$$

Algebra morphism

$$
\begin{tikzcd}
TTC \arrow[r, "TTf"] \arrow[d, "\mu^T_C"'] & TTA \arrow[r, "Ta"] \arrow[d, "\mu^T_A"] & TA \arrow[d, "a"] \\
TC \arrow[r, "Tf"'] & TA \arrow[r, "a"'] & A
\end{tikzcd}
$$

Uniqueness

$$\begin{tikzcd}
TC \arrow[r, "T\eta^T_C"] \arrow[rr, bend left, "Tf"] \arrow[dr, equal] & TTC \arrow[r, "Tf'"] \arrow[d, "\mu^T_C"'] & TA \arrow[d, "a"] \\
& TC \arrow[r, "f'"'] & A
\end{tikzcd}$$

Universal property of free algebra

We have proved:

Proposition

For any T-algebra $a\colon TA \to A$, the map

$$T\text{-}\mathbf{Alg}(TC, A) \xrightarrow{\mathbf{U}^T_{TC,A}} \mathbf{C}(TC, A) \xrightarrow{\mathbf{C}(\eta^T_C, A)} \mathbf{C}(C, A)$$

is bijective.

$$\begin{tikzcd} C \arrow[r, "\eta^T_C"] \arrow[dr, "f"'] & TC \arrow[d, dashed, "\tilde{f}"] \\ & A \end{tikzcd}$$

Generalisation: adjunctions

Replace

$$\mathbf{U}^T : T\text{-}\mathbf{Alg} \to \mathbf{C}$$

with arbitrary

$$U \colon \mathbf{A} \to \mathbf{C}.$$

Generalisation: adjunctions

Definition (*Adjunction*)

Functor $U\colon \mathbf{A} \to \mathbf{C}$ equipped with

- $F_0(C)$ for all C and
- $\eta_C\colon C \to UF_0C$

such that for all $A \in \mathbf{A}$ and f as in

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & UF_0C \arrow[d, "U\tilde{f}"] \\ & UA \end{tikzcd}$$

there exists a unique $\tilde{f}\colon F_0C \to A$ making the triangle commute.

Algebras $\rightsquigarrow$ adjunction

- $F_0(C)$ is (TC, μ^T_C),
- $\eta_C : C \to UF_0C$ is $\eta^T_C : C \to TC$.

Symmetrisation

Proposition

For any adjunction $U : \mathbf{A} \to \mathbf{C}$,

- F_0 extends to a unique functor $F : \mathbf{C} \to \mathbf{A}$
- making $\eta : id \to UF$ natural.

Proof.

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] & UF_0C \arrow[d, "U\widetilde{\eta_D \circ f}"] \\ D \arrow[r, "\eta_D"] & UF_0D \end{tikzcd} \qquad \begin{tikzcd} F_0C \arrow[d, "F(f) := \widetilde{\eta_D \circ f}"] \\ F_0D \end{tikzcd}$$

□

Symmetrisation

Corollary

Adjunctions (U, F_0, η) are in 1-1 correspondence with pairs

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] & & \mathbf{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

equipped with natural $\eta \colon id_{\mathbf{C}} \to UF$, such that

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & UFC \arrow[d, "U\tilde{f}"] \\ & UA \end{tikzcd}$$

as before.

Symmetrisation

Definition

We call triples (U, F, η) *functorial adjunctions*.

Symmetrisation continued

Proposition

Any adjunction (U, F_0, η) gives rise to a unique $\varepsilon \colon FU \longrightarrow id_{\mathbf{A}}$ such that

$$\begin{tikzcd} UA \arrow[r, "\eta_{UA}"] \arrow[dr, equal] & UFUA \arrow[d, "U\varepsilon_A"] \\ & UA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, equal] & FUFC \arrow[d, "\varepsilon_{FC}"] \\ & FC \end{tikzcd}$$

Symmetrisation continued

$$\begin{tikzcd} UA \arrow[r, "\eta_{UA}"] \arrow[dr, equal] & UFUA \arrow[d, "U\varepsilon_A"] \\ & UA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, equal] & FUFC \arrow[d, "\varepsilon_{FC}"] \\ & FC \end{tikzcd}$$

Proof.

Define it as below right.

$$\begin{tikzcd} UA \arrow[r, "\eta_{UA}"] \arrow[dr, equal] & UFUA \arrow[d, "U\widetilde{id_{UA}}"] \\ & UA \end{tikzcd} \qquad \begin{tikzcd} FUA \arrow[d, "\varepsilon_A := \widetilde{id_{UA}}"] \\ A \end{tikzcd}$$

Entails left-hand triangle commutes.

□

Symmetrisation continued

$$\begin{tikzcd} UA \arrow[r, "\eta_{UA}"] \arrow[dr, equal] & UFUA \arrow[d, "U\varepsilon_A"] \\ & UA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, equal] & FUFC \arrow[d, "\varepsilon_{FC}"] \\ & FC \end{tikzcd}$$

Proof.

Uniqueness at $A := FC$ gives right-hand triangle.

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"] & UFC \arrow[d, "UF\eta_C"'] \arrow[dd, bend left=60, equal] \\ UFC \arrow[r, "\eta_{UFC}"] \arrow[dr, bend right, equal] & UFUFC \arrow[d, "U\widetilde{id_{UFC}}"'] \\ & UFC \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[d, "F\eta_C"] \\ FUFC \arrow[d, "\varepsilon_{FC} := \widetilde{id_{UFC}}"] \\ FC \end{tikzcd}$$

□

Symmetrisation continued

Corollary

Adjunctions (U, F_0, η) *are in 1-1 correspondence with pairs*

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] & & \mathbf{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

equipped with natural η *and* ε*, such that*

$$\begin{tikzcd} UA \arrow[r, "\eta_{UA}"] \arrow[dr, equal] & UFUA \arrow[d, "U\varepsilon_A"] \\ & UA \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, equal] & FUFC \arrow[d, "\varepsilon_{FC}"] \\ & FC \end{tikzcd}$$

Symmetrisation continued

Proof.

Remains to prove that any such $(U, F, \eta, \varepsilon)$ yields an adjunction.
Existence of $\widetilde{f}$:

$$
\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dd, "f"'] & UFC \arrow[d, "UFf"] & & FC \arrow[dd, "\widetilde{f} := \varepsilon_A \circ Ff"] \\
& UFUA \arrow[d, "U\varepsilon_A"] & & \\
UA \arrow[ur, "\eta_{UA}"] \arrow[r, equal] & UA & & A
\end{tikzcd}
$$

$\square$

Symmetrisation continued

Proof.

Uniqueness of $\widetilde{f}$:

$$\begin{tikzcd}
FC \arrow[r, "F\eta_C"] \arrow[rr, bend left, equal] \arrow[dr, "Ff"'] & FUFC \arrow[r, "\varepsilon_{FC}"] \arrow[d, "FUf'"] & FC \arrow[d, "f'"] \\
& FUA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}$$

□

Symmetrisation continued

Definition

We call such 4-tuples $(U, F, \eta, \varepsilon)$ *balanced adjunctions*.

Symmetrisation III

Adjunctions (U, F_0, η) are "rightist": they emphasise U, which is called the *right adjoint*.

Definition (*leftist adjunction*)

Functor $F \colon \mathbf{C} \to \mathbf{A}$ equipped with

- $U_0(A)$ for each A and
- $\varepsilon_A \colon FU_0A \to A$

such that for all C and f as in

$$
\begin{tikzcd}
U_0A & FU_0A \arrow[r, "\varepsilon_A"] & A \\
C \arrow[u, "\tilde{f}"] & FC \arrow[u, "F\tilde{f}"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

there exists a unique $\tilde{f} \colon C \to U_0A$ making the triangle commute.

Symmetrisation III

Proposition

For any leftist adjunction (F, U_0, ε), U extends to a unique functor making ε natural.

Proof.

Similar to the rightist case. $\square$

Symmetrisation III

Corollary

Leftist adjunctions are in 1-1 correspondence with pairs

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] & & \mathbf{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

equipped with natural $\varepsilon \colon FU \to id_{\mathbf{A}}$*, such that*

$$\begin{tikzcd} U_0A & & FU_0A \arrow[r, "\varepsilon_A"] & A \\ C \arrow[u, "\exists! \tilde{f}"] & & FC \arrow[u, "F\tilde{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

as before.

Symmetrisation III

Definition

We call triples (U, F, ε) *functorial leftist adjunctions*.

Symmetrisation IV

Corollary

Functorial leftist adjunctions (U, F, ε) are in 1-1 correspondence with balanced adjunctions.

Proof.

Similar to rightist case. □

Symmetrisation V

Proposition

Given any adjunction the properties

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & UFC \arrow[d, "U\tilde{f}"] \\ & UA \end{tikzcd} \quad \begin{tikzcd} FC \arrow[d, "\tilde{f}"] \\ A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} UA \\ C \arrow[u, "\tilde{f}"] \end{tikzcd} \quad \begin{tikzcd} FUA \arrow[r, "\varepsilon_A"] & A \\ FC \arrow[u, "F\tilde{f}"] \arrow[ur, "f"'] \end{tikzcd}$$

induce a bijection

$$\mathbf{C}(C, UA) \cong \mathbf{A}(FC, A)$$

which is a natural isomorphism

$$\begin{tikzcd} & \mathbf{C}^{op} \times \mathbf{C} \arrow[dr, "\mathbf{C}(-_1, -_2)"] & \\ \mathbf{C}^{op} \times \mathbf{A} \arrow[ur, "\mathbf{C}^{op} \times U"] \arrow[dr, "F^{op} \times \mathbf{A}"'] & \cong & \mathbf{Set}. \\ & \mathbf{A}^{op} \times \mathbf{A} \arrow[ur, "\mathbf{A}(-_1, -_2)"'] & \end{tikzcd}$$

Preparatory exercises

1. Equip the product graph $\mathbf{C} \times \mathbf{D}$ with category structure, making it a product in $\mathbf{Cat}$.
2. A transformation α between functors $F, G \colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is natural iff it is natural in each component, i.e.,

   $$\alpha_{-,D} \colon F(-, D) \to G(-, D) \qquad \text{and} \qquad \alpha_{C,-} \colon F(C, -) \to G(C, -)$$

   are both natural.
3. A natural transformation $\alpha \colon F \to G \colon \mathbf{C} \to \mathbf{D}$ whose components $\alpha_C \colon FC \to GC$ are isomorphisms, is a natural isomorphism.

Naturality proof

$$C \xrightarrow{f} UA \qquad \longmapsto \qquad FC \xrightarrow{\widetilde{f}} A$$

$$\begin{tikzcd} C & A \arrow[d, "v"] \\ C' \arrow[u, "u"] & A' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C}(C,UA) \arrow[r, "\cong"] \arrow[d, "\mathbf{C}(u,Uv)"'] & \mathbf{A}(FC,A) \arrow[d, "\mathbf{A}(Fu,v)"] \\ \mathbf{C}(C',UA') \arrow[r, "\cong"'] & \mathbf{A}(FC',A') \end{tikzcd}$$

$$\begin{tikzcd} FC' \arrow[r, "Fu"'] & FC \arrow[r, "\widetilde{f}"'] & A \arrow[r, "v"'] & A' \end{tikzcd}$$

$$\| ?$$

$$C' \xrightarrow{u} C \xrightarrow{f} UA \xrightarrow{Uv} UA' \qquad \longmapsto \qquad \widetilde{U(v) \circ f \circ u}$$

It suffices to check that the candidate $v \circ \widetilde{f} \circ Fu$ satisfies the universal property of $\widetilde{U(v) \circ f \circ u}$.

Naturality proof

$$
\begin{tikzcd}
C' \arrow[rrr, "\eta_{C'}"] \arrow[dr, "u"'] & & & UFC' \arrow[d, "UFu"] & & FC' \arrow[d, "Fu"] \\
& C \arrow[rr, "\eta_C"] \arrow[dr, "f"'] & & UFC \arrow[d, "U\widetilde{f}"] & & FC \arrow[d, "\widetilde{f}"] \\
& & UA \arrow[r, equal] \arrow[dr, "Uv"'] & UA \arrow[d, "Uv"] & & A \arrow[d, "v"] \\
& & & UA' & & A'
\end{tikzcd}
$$

Symmetrisation V

Corollary

(Rightist functorial) adjunctions are in 1-1 correspondence with pairs

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] & & \mathbf{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

equipped with a natural isomorphism

$$\mathbf{C}(C, UA) \cong \mathbf{A}(FC, A).$$

Symmetrisation V: proof

$$\begin{array}{llcl} & \mathbf{C}(C, UA) & \cong & \mathbf{A}(FC, A) \\ \text{define:} & \eta_C & \leftarrow\!\shortmid & id_{FC}\ (A = FC) \\ (C = UA) & id_{UA} & \mapsto & \varepsilon_A \end{array}$$

then have:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\widetilde{f'}"'] & UFC \arrow[d, "Uf'"] \\ & UA \end{tikzcd} \quad \leftarrow\!\shortmid \quad \begin{tikzcd} FC \arrow[d, "f'"] \\ A \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[d, "f"'] \\ UA \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} FC \arrow[d, "Ff"'] \arrow[dr, "\widetilde{f}"] & \\ FUA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

Symmetrisation V: proof

Indeed:

$$
\begin{tikzcd}
UA \arrow[r, equal] \arrow[ddd, mapsto] & UA \arrow[rr, mapsto] & & FUA \arrow[r, "\varepsilon_A"] & A \arrow[ddd, mapsto] \\
& \mathbf{C}(UA, UA) \arrow[r, "\cong"'] \arrow[d, "\mathbf{C}(f, UA)"'] & \mathbf{A}(FUA, A) \arrow[d, "\mathbf{A}(Ff, A)"] & & \\
& \mathbf{C}(C, UA) \arrow[r, "\cong"] & \mathbf{A}(Ff, A) & & \\
C \arrow[r, "f"'] & UA \arrow[rr, mapsto] & & FUA \arrow[r, "\varepsilon_A"] & A \\
& & & FC \arrow[u, "Ff"] \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}
$$

Symmetrisation V: proof

Indeed:

$$
\begin{tikzcd}
FC \arrow[r, equal] \arrow[ddd, mapsto] & FC \arrow[rrr, mapsto] & & & C \arrow[r, "\eta_C"] & UFC \arrow[dd, mapsto] \\
& & \mathbf{A}(FC,FC) \arrow[r, "\cong"'] \arrow[d, "\mathbf{A}(FC,f')"'] & \mathbf{C}(C,UFC) \arrow[d, "\mathbf{C}(C,Uf')"] & & \\
& & \mathbf{A}(FC,A) \arrow[r, "\cong"] & \mathbf{C}(C,UA) & & UFC \arrow[d, "Uf'"] \\
FC \arrow[r, "f'"] & A \arrow[rrr, mapsto] & & & C \arrow[ur, "\eta_C"] \arrow[r, "\widetilde{f'}"'] & UA
\end{tikzcd}
$$

Symmetrisation V: proof

Need to check universal property of $\widetilde{f}$, for $f : C \longrightarrow UA$.

- Commutation: $U(\widetilde{f}) \circ \eta_C = \widetilde{\widetilde{f}} = f$.
- Uniqueness:
  - if $U(f') \circ \eta_C = f$,
  - i.e., $\widetilde{f'} = f$,
  - then apply $\widetilde{(-)}$ to get $f' = \widetilde{f}$.

Symmetrisation V

Definition

We call *hom-based adjunction* pairs

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] && \mathbf{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

equipped with a natural isomorphism

$$\mathbf{C}(C, UA) \cong \mathbf{A}(FC, A).$$

Symmetrisation summary

1-1 correspondences between:

Name	Data	Properties
Rightist adjunction	(U, F_0, η)	$f \mapsto \widetilde{f}$
Rightist functorial adjunction	(U, F, η)	$f \mapsto \widetilde{f}$
Balanced adjunction	$(U, F, \eta, \varepsilon)$	zig-zag "$\varepsilon \circ \eta = id$"
Hom-based adjunction	$(U, F, \widetilde{-})$	natural iso
Leftist functorial adjunction	(F, U, ε)	$f' \mapsto \widetilde{f'}$
Leftist adjunction	(F, U_0, ε)	$f' \mapsto \widetilde{f'}$

Symmetrisation summary

Notation

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathbf{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

- F is called the *left* adjoint.
- U is called the *right* adjoint.
- ($\vdash$ symbol points to left adjoint)

Outline

Interlude II: Kleisli

Fix monad $T\colon \mathbf{C} \to \mathbf{C}$.
We have seen T-$\mathbf{Alg} \to \mathbf{C}$ yields an adjunction.
We saw:

- Unit $\eta^T_C\colon C \to TC$.
- Transpose $f\colon C \to A$ to $TC \xrightarrow{Tf} TA \xrightarrow{a} A$.

We haven't seen:

- Counit $\varepsilon^T_A\colon TA \to A$ given by algebra structure a.
- Transpose $f\colon TC \to A$ as $C \xrightarrow{\eta^T_C} TC \xrightarrow{f} A$.

Exercise

Algebra structures and forgetful functor $\mathbf{U}^T$ are implicit here: reformulate this slide in excruciating detail.

Another adjunction derived from any monad T

Notation

Let $\mathbf{F}^T : \mathbf{C} \to T\text{-}\mathbf{Alg}$ denote the left adjoint.

Definition

Identity-on-objects / fully faithful factorisation

$$\mathbf{C} \xrightarrow{\mathbf{io}_T} \mathbf{Kl}^T \xrightarrow{\mathbf{ff}_T} T\text{-}\mathbf{Alg}.$$

- Objects: those of $\mathbf{C}$.
- Morphism $C \to D$: algebra morphism $TC \to TD$.

Another adjunction derived from any monad T

Proposition

Adjunction

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "\mathbf{io}^T"] \arrow[rr, phantom, "\perp"] & & \mathbf{Kl}^T \arrow[dl, bend left, "\mathbf{ff}^T"] \\ & T\text{-}\mathbf{Alg} \arrow[ul, bend left, "\mathbf{U}^T"] & \end{tikzcd}$$

Proof

We prove the more general

Lemma

For any fully faithful F:

$$
\begin{tikzcd}
 & \mathbf{K} \arrow[dr, "F"] & \\
\mathbf{C} \arrow[ur, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{A} \arrow[ll, bend left, "U"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\mathbf{C} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{K} \arrow[dl, "F"] \\
 & \mathbf{A} \arrow[ul, "U"] &
\end{tikzcd}
$$

$$
\begin{aligned}
\mathbf{C}(C, UFK) &\cong \mathbf{A}(FLC, FK) && \text{(by adjunction)} \\
&\cong \mathbf{K}(LC, K) && \text{(by full faithfulness).}
\end{aligned}
$$

Both adjunctions in one picture

$$\begin{tikzcd}
\mathbf{Kl}^T \arrow[rr, bend left=15, "\mathbf{ff}^T"] \arrow[dr, bend left, "\mathbf{U}^T\mathbf{ff}^T"] & & T\text{-}\mathbf{Alg} \arrow[dl, bend left, "\mathbf{U}^T"] \\
& \mathbf{C} \arrow[ul, bend left, "\mathbf{io}^T"] \arrow[ul, phantom, "\perp"] \arrow[ur, bend left, "\mathbf{F}^T"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

Remark, in passing

Definition (*Resolution* of monad T)

Adjunction $F \dashv U$ such that $UF = T$.

Form a category $\mathbf{Res}(T)$ with morphisms given by H making

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, bend left, "H"] \arrow[dr, bend left, "U"] & & \mathbf{B} \arrow[dl, bend left, "V"] \\
& \mathbf{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "G"] \arrow[ul, phantom, "\dashv"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
$$

commute.

Remark, in passing

Proposition

- $\mathbf{F}^T \dashv \mathbf{U}^T$ is terminal in $\mathbf{Res}(T)$.
- $\mathbf{io}^T \dashv \mathbf{U}^T \mathbf{ff}^T$ is initial in $\mathbf{Res}(T)$.

Back on track

By definition and adjunction, we have a bijection

$$\begin{aligned}
\mathbf{Kl}^T(C,D) &= \mathbf{Kl}^T(\mathbf{io}^T C, \mathbf{io}^T D) \\
&\cong T\text{-}\mathbf{Alg}(\mathbf{ff}^T \mathbf{io}^T C, \mathbf{ff}^T \mathbf{io}^T D) \\
&= T\text{-}\mathbf{Alg}(\mathbf{F}^T C, \mathbf{F}^T D) \\
&\cong \mathbf{C}(C, \mathbf{U}^T \mathbf{F}^T D) \\
&\cong \mathbf{C}(C, TD) \\
(f : TC \to TD) &\mapsto f \circ \eta^T_C.
\end{aligned}$$

Back on track

Notation

We write $\to\!\!\!\!\!\bullet$ for arrows in $\mathbf{Kl}^T$.

Proposition (Characterisation of the Kleisli cat)

Across the bijections $\mathbf{Kl}^T(C, D) \cong \mathbf{C}(C, TD)$:

- Identity $C \to\!\!\!\!\!\bullet\; C$ becomes $\eta^T_C : C \to TC$.
- Composition $C \xrightarrow{f}\!\!\!\!\!\bullet\; D \xrightarrow{g}\!\!\!\!\!\bullet\; E$ becomes
  $C \xrightarrow{f} TD \xrightarrow{Tg} TTE \xrightarrow{\mu^T_E} TE$.

Remark

Probably the most frequently used presentation of $\mathbf{Kl}^T$.

Proof

Direct for identity. Here's composition:

$$
\begin{array}{c}
\mathbf{C}(C,TD) \times \mathbf{C}(D,TE) \\
\downarrow \cong \\
\mathbf{Kl}^T(C,D) \times \mathbf{Kl}^T(D,E) \\
\downarrow \circ \\
\mathbf{Kl}^T(C,E) \\
\downarrow \cong \\
\mathbf{C}(C,TE)
\end{array}
$$

$$C \xrightarrow{f} TD \qquad D \xrightarrow{g} TE$$

$$TC \xrightarrow{\mu^T_D \circ Tf} TD \qquad TD \xrightarrow{\mu^T_E \circ Tg} TE$$

$$TC \xrightarrow{\mu^T_D \circ Tf} TD \xrightarrow{\mu^T_E \circ Tg} TE$$

$$
\begin{tikzcd}
C \arrow[r, "\eta^T_C"] \arrow[dr, "f"'] & TC \arrow[r, "Tf"] & TTD \arrow[r, "\mu^T_D"] & TD \arrow[d, "Tg"] \\
& TD \arrow[ur, "\eta^T_{TD}"] \arrow[urr, equal] \arrow[rr, "Tg"'] & & TTE \arrow[d, "\mu^T_E"] \\
& & & TE
\end{tikzcd}
$$

Outline

Lawvere theories from monads

Let us revisit the syntactic category $\mathbb{L}_\Sigma$, and then generalise.

- Define monad Σ^*.
- Observe $\mathbb{L}_\Sigma \hookrightarrow (\mathbf{Kl}^{\Sigma^*})^{op}$ as the full subcategory $\mathbf{Kl}^{\Sigma^*}_{\mathbb{F}}$ spanned by finite cardinals.
- Generalise: for any monad T, $\mathbf{Kl}^{T}_{\mathbb{F}}$ is a Lawvere theory.

Main idea

Let $T\colon \mathbf{Set} \to \mathbf{Set}$ be any monad.

- $T(X)$ thought of as set of terms with free variables in X.
- $n \to T(m)$: n-tuples of terms with m free variables.
- Exactly the idea behind morphisms $m \to n$ in syntactic categories of shape $\mathbb{L}_\Sigma$.

Let us elaborate.

Free monads

Fix an algebraic signature Σ given by $a \colon O \to \mathbb{N}$.
Recall Σ-terms $\Sigma^*(X) = \{X \vdash_\Sigma e\}$:

$$\frac{\textsc{Var}}{X \vdash_\Sigma [x]}\ (x \in X) \qquad \frac{\overset{\textsc{Op}}{X \vdash_\Sigma e_1 \quad \ldots \quad X \vdash_\Sigma e_p}}{X \vdash_\Sigma o(e_1, \ldots, e_p)}\ (a(o) = p)$$

- By definition:

$$\begin{aligned} \mathbb{L}_\Sigma(m,n) &= \Sigma^*(m)^n \\ &= \mathbf{Set}(n, \Sigma^*(m)) \\ &\cong ?\ \mathbf{Kl}^{\Sigma^*}(n,m). \end{aligned}$$

Need Σ^* to be a monad.

Functor structure of Σ^*

Σ^* extends to a functor $\mathbf{Set} \to \mathbf{Set}$.
For any $f \colon X \to Y$, let

$$\begin{aligned} \Sigma^*(f) \colon \Sigma^*(X) &\to \Sigma^*(Y) \\ [x] &\mapsto [f(x)] \\ o(e_1, \ldots, e_p) &\mapsto o(\Sigma^*(f)(e_1), \ldots, \Sigma^*(f)(e_p)). \end{aligned}$$

"Rename variables according to f."

Monad structure of Σ^*

- Unit $[-] \colon X \to \Sigma^*(X)$
- Multiplication $\Sigma^*(\Sigma^*(X)) \to \Sigma^*(X)$.
  - Elements of $\Sigma^*(\Sigma^*(X))$: brackets contain terms.
  - Multiplication: remove outer brackets.
  - Example: $o(o([o([x],[y])],[[x]]),[o([y],[y])])$.
- Monad equations ($T = \Sigma^*$) reminder.

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, equal] & T(T(X)) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, equal] \\
& T(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\
T(T(X)) \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

Exercise

Check them (not necessarily too formally).

Monad structure of Σ^*

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, equal, bend right] & T(T(X)) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, equal, bend left] \\ & T(X) & \end{tikzcd} \qquad \begin{tikzcd} T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\ T(T(X)) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

- Remove outer brackets in $[e]$ yields e.
- Replace each $[x]$ with $[[x]]$ in e, then remove outer brackets yields e.
- Given term with three bracket layers:
  - remove outer, then middle, and
  - remove middle, then outer

  agree.

Syntactic vs Kleisli

Definition

Let $\mathbf{Kl}^{\Sigma^*}_{\mathbb{F}} \hookrightarrow \mathbf{Kl}^{\Sigma^*}$ denote the full subcategory spanned by finite cardinals.

Proposition

$\mathbb{L}_\Sigma \cong (\mathbf{Kl}^{\Sigma^*}_{\mathbb{F}})^{op}$.

"Syntactic category = op-Kleisli restricted to $\mathbb{F}$."

Syntactic vs Kleisli

Proposition

$\mathbb{L}_\Sigma \cong (\mathbf{Kl}_{\mathbb{F}}^{\Sigma^*})^{op}$.

Proof.

We have proved the graphs agree.

- Identities: $\eta_n : n \to \Sigma^*(n)$ does correspond to $\langle [1], \ldots, [n] \rangle$.
- Composition $m \xrightarrow{f} \Sigma^*(n)$, $n \xrightarrow{g} \Sigma^*(p)$.
  - By substitution in $\mathbb{L}_\Sigma$.
  - In the Kleisli, jth term is
    - $f(j)$,
    - with each $[i]$ replaced with $g(i)$, i.e.,...

  substitution. □

Generalisation

Proposition

For any monad $T\colon \mathbf{Set} \to \mathbf{Set}$, $(\mathbf{Kl}^T_{\mathbb{F}})^{op}$ is a Lawvere theory.

Proof.

- Small: ✓.
- Skeletal: ✓.
- Finite products:
  - $\mathbb{F}$ has finite coproducts and
  - $\mathbb{F} \hookrightarrow \mathbf{Kl}^T$ preserves them (exercise!);
  - so does $\mathbb{F} \hookrightarrow \mathbf{Kl}^T_{\mathbb{F}}$ by full faithfulness,
  - hence $\mathbf{Kl}^T_{\mathbb{F}}$ has finite coproducts.
- Power generation: every n is $\underbrace{1 + \ldots + 1}_{n \text{ times}}$. □

Outline

Monads from Lawvere theories

Let us now sketch the other direction:

> monads $\longrightarrow$ Lawvere theories

saving functoriality of the correspondence for later.

Starting point

Let us fix a Lawvere theory $\mathbb{L}$.
Idea:

$$\begin{aligned} \mathbb{F} &\to \mathbf{Set} \\ n &\mapsto \mathbb{L}(n, 1). \end{aligned}$$

How to extend this to arbitrary sets?

A first extension

Definition

$\mathbf{T}^0_{\mathbb{L}}(X) = \coprod_{n \in \mathbb{F}} \mathbb{L}(n, 1) \times X^n$.

"A term with interpretation of variables in X".

Problem

Too fine: two distinct ways of representing $o(x, x)$.

$$\begin{tikzcd}
& 1 & \\
2 \arrow[ur, "o"] \arrow[dr, "{[x,x]}"'] & & 1 \arrow[ul, "o\Delta"'] \arrow[dl, "x"] \\
& X &
\end{tikzcd}$$

$(2, o, (x, x))$ vs $(1, o\Delta, (x))$.

(Here, $\Delta \colon 1 \to 2$ is $\langle id_1, id_1 \rangle$.)

Solution: quotient out

In the example:

$$
\begin{tikzcd}
 & 1 & \\
2 \arrow[ur, "o"] & & 1 \arrow[ul, "o\Delta"'] \arrow[ll, "\Delta"'] \\
2 \arrow[rr, "!"'] \arrow[dr, "{[x,x]}"'] & & 1 \arrow[dl, "x"] \\
 & X &
\end{tikzcd}
$$

Solution: quotient out

In general, let

$$(m, h, v) \sim (n, k, w)$$

for any $f : m \to n$ making the following commute.

$$\begin{tikzcd}
 & 1 & \\
m \arrow[ur, "h"] & & n \arrow[ul, "k"'] \arrow[ll, "f"'] \\
m \arrow[rr, "f"'] \arrow[dr, "v"'] & & n \arrow[dl, "w"] \\
 & X &
\end{tikzcd}$$

Otherwise said

For all, $X \xleftarrow{w} n \xleftarrow{f} m \xrightarrow{h} 1$, $\qquad (m, h, wf) \sim (n, hf, w)$.

Your first coend

Definition

Let $\mathbf{T}_{\mathbb{L}}(X) = (\coprod_n \mathbb{L}(n,1) \times X^n)/\sim$.

Remark

- *Standard notation* $\int^n \mathbb{L}(n,1) \times X^n$.
- *Called a coend.*
- *Satisfies universal property.*

$$
\begin{tikzcd}
\mathbb{F} \arrow[rr, hook] \arrow[dr, "\mathbb{L}(-,1)"'] & \arrow[d, phantom, "\stackrel{\lambda_{\mathbb{L}}}{\Rightarrow}"] & \mathbf{Set} \arrow[dl, "\mathbf{T}_{\mathbb{L}}"] \\
& \mathbf{Set} &
\end{tikzcd}
$$

Next goal

Show that $\mathbf{T}_{\mathbb{L}}$ is indeed a monad.

Without all technical detail.

Outline

First step: Kleisli presentation of monads

Definition (*hom-based monad* on category $\mathbf{C}$)

- Object assignement $T : \mathbf{ob}(\mathbf{C}) \to \mathbf{ob}(\mathbf{C})$.
- *Unit* $\eta_C : C \to TC$.
- *Kleisli lifting*: $\mathbf{C}(C, TD) \to \mathbf{C}(TC, TD)$.
- Axioms:

$$\begin{tikzcd}
TC \arrow[r, bend left, "\eta_C^\dagger"] \arrow[r, bend right, equal] & TC
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[rr, "\eta_C"] \arrow[dr, "f"'] & & TC \arrow[dl, "f^\dagger"] \\
& TD &
\end{tikzcd}
\qquad
\begin{tikzcd}
TC \arrow[rr, "f^\dagger"] \arrow[dr, "(g^\dagger \circ f)^\dagger"'] & & TD \arrow[dl, "g^\dagger"] \\
& TE &
\end{tikzcd}$$

Monads to hom-based monads

Proposition

Every monad T yields a hom-based monad with

- obvious object assignment $T : \mathbf{ob}(\mathbf{C}) \to \mathbf{ob}(\mathbf{C})$,
- obvious unit $\eta^T_C : C \to TC$,
- Kleisli lifting

$$\begin{array}{ccc} \mathbf{C}(C, TD) & \to & \mathbf{C}(TC, TD) \\ C \xrightarrow{f} TD & \mapsto & TC \xrightarrow{Tf} TTD \xrightarrow{\mu^T_D} TD. \end{array}$$

Exercise: check the axioms

$$\begin{tikzcd} TC \arrow[rr, bend left, "\eta^\dagger_C"] \arrow[rr, bend right, equal] & & TC \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\eta_C"] \arrow[dr, "f"'] & & TC \arrow[dl, "f^\dagger"] \\ & TD & \end{tikzcd} \qquad \begin{tikzcd} TC \arrow[rr, "f^\dagger"] \arrow[dr, "(g^\dagger \circ f)^\dagger"'] & & TD \arrow[dl, "g^\dagger"] \\ & TE & \end{tikzcd}$$

Axiom 1

$$\begin{tikzcd}
TA \arrow[r, bend left, "T\eta^T_A"] \arrow[rr, bend right, equal] & TTA \arrow[r, bend left, "\mu^T_A"] & TA
\end{tikzcd}$$

Axiom 2

$$
\begin{tikzcd}
C \arrow[r, "\eta^T_C"] \arrow[d, "f"'] & TC \arrow[d, "Tf"] \\
TD \arrow[r, "\eta^T_{TD}"] \arrow[dr, equal, bend right] & TTD \arrow[d, "\mu^T_D"] \\
 & TD
\end{tikzcd}
$$

Axiom 3

$$
\begin{tikzcd}
TC \arrow[r, "Tf"] \arrow[dd, "T(g^\dagger \circ f)"'] & TTD \arrow[r, "\mu^T_D"] \arrow[d, "TTg"] & TD \arrow[d, "Tg"] \\
 & TTTE \arrow[r, "\mu^T_{TE}"] \arrow[dl, "T(\mu^T_E)"] & TTE \arrow[d, "\mu^T_E"] \\
TTE \arrow[rr, "\mu^T_E"] & & TE
\end{tikzcd}
$$

Converse

Proposition

Every hom-based monad T yields a proper monad with

- obvious object assignment $T\colon \mathbf{ob}(\mathbf{C}) \to \mathbf{ob}(\mathbf{C})$,
- morphism assignment
$$\begin{array}{rcl} \mathbf{C}(C,D) & \to & \mathbf{C}(TC,TD) \\ C \xrightarrow{f} D & \mapsto & (\eta_D \circ f)^\dagger, \end{array}$$
- obvious unit $\eta^T_C\colon C \to TC$,
- multiplication μ_C given by $id^\dagger_{TC}$.

Exercise: check the axioms

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, equal] & T(T(X)) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, equal] \\ & T(X) & \end{tikzcd}
\qquad
\begin{tikzcd} T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\ T(T(X)) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

Axiom 1

$$\begin{tikzcd}
T(C) \arrow[r, "\eta_{T(C)}"] \arrow[dr, equal, bend right] & T(T(C)) \arrow[d, "id^{\dagger}_{TC}"] & T(C) \arrow[l, "T(\eta_C)"'] \arrow[dl, equal, bend left] \\
& T(C) &
\end{tikzcd}$$

Axiom 2

$$
\begin{tikzcd}
T(T(C)) \arrow[d, "id^\dagger_{TC}"'] & & T(C) \arrow[ll, "(\eta_{TC} \circ \eta_C)^\dagger"'] \arrow[dll, bend right=15, "(id^\dagger_{TC} \circ \eta_{TC} \circ \eta_C)^\dagger"'] \arrow[dll, bend left=20, "\eta^\dagger_C"] \arrow[dll, equal, bend left=70] \\
T(C) & &
\end{tikzcd}
$$

Axiom 3

$$
\begin{tikzcd}
T(T(T(X))) \arrow[rr, "(\eta_{TX} \circ id^{\dagger}_{TX})^{\dagger}"] \arrow[d, "id^{\dagger}_{TTX}"'] \arrow[drr, bend right, "(id^{\dagger}_{TX} \circ id_{TTX})^{\dagger}"'] \arrow[drr, "id^{\dagger\dagger}_{TX}"] \arrow[drr, bend left, "(id^{\dagger}_{TX} \circ \eta_{TX} \circ id^{\dagger}_{TX})^{\dagger}"] & & T(T(X)) \arrow[d, "id^{\dagger}_{TX}"] \\
T(T(X)) \arrow[rr, "id^{\dagger}_{TX}"'] & & T(X)
\end{tikzcd}
$$

Outline

Next goal

Show that $\mathbf{T}_{\mathbb{L}}$ is indeed a hom-based monad.

Without all technical detail.

Kleisli lifting

$$\mathbf{Set}(X, \mathbf{T}_{\mathbb{L}} Y) \to \mathbf{Set}(\mathbf{T}_{\mathbb{L}} X, \mathbf{T}_{\mathbb{L}} Y)$$

- Let $\sigma \colon X \to \mathbf{T}_{\mathbb{L}} Y$ and $e \in \mathbf{T}_{\mathbb{L}} X$.
- Pick representative $X \xleftarrow{v} n \xrightarrow{h} 1$ for e.
- Only the composite $n \xrightarrow{v} X \xrightarrow{\sigma} \mathbf{T}_{\mathbb{L}} Y$ will matter.
- Pick representative of e'_i for each $\sigma v(i)$, $i \in n$, say

$$Y \xleftarrow{v'_i} p_i \xrightarrow{h'_i} 1.$$

- Return

$$Y \xleftarrow{[v'_i]_i} \sum_i p_i \xrightarrow{\langle h'_i \circ \pi_i \rangle} n \xrightarrow{h} 1.$$

Detail omitted

This forms a hom-based monad.
Upshot: unitality and associativity boil down to unitality and associativity of composition in $\mathbb{L}$.

Outline

③ Linton's theorem
- Introduction
- Syntactic categories
- Lawvere theories and their models
- Interlude I: adjunctions
- Interlude II: Kleisli category
- Lawvere theories from monads
- Monads from Lawvere theories
- Interlude: hom-based monads
- Lawvere theories to monads, continued
- Functoriality

④ Grothendieck's nerve theorem and the Segal condition

⑤ Sketching a general correspondence between monads and theories

Category of monads

Definition ($\mathbf{Mnd(C)}$)

- Objects: monads on $\mathbf{C}$.
- Morphism $(T, \mu, \eta) \to (T', \mu', \eta')$: any $\alpha : T \to T'$ making the following commute.

$$
\begin{tikzcd}
 & id_{\mathbf{C}} \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\
T \arrow[rr, "\alpha"'] & & T'
\end{tikzcd}
\qquad
\begin{tikzcd}
TT \arrow[r, "\alpha \circ_0 \alpha"] \arrow[d, "\mu"'] & T'T' \arrow[d, "\mu'"] \\
T \arrow[r, "\alpha"'] & T'
\end{tikzcd}
$$

Functorial action $\mathbf{Mnd}(\mathbf{Set}) \to \mathbf{Law}$

Proposition

Monad morphism $\alpha\colon (T, \mu, \eta) \to (T', \mu', \eta')$
$\rightsquigarrow$ Lawvere theory morphism

$$\begin{aligned} \mathbf{Kl}^{\alpha}_{\mathbb{F}}\colon \mathbf{Kl}^{T}_{\mathbb{F}} &\to \mathbf{Kl}^{T'}_{\mathbb{F}} \\ n &\mapsto n \\ (n \xrightarrow{f} Tp) &\mapsto (n \xrightarrow{f} Tp \xrightarrow{\alpha_p} T'p). \end{aligned}$$

Proof: functoriality of $\mathbf{Kl}^{\alpha}_{\mathbb{F}} \colon \mathbf{Kl}^{T}_{\mathbb{F}} \to \mathbf{Kl}^{T'}_{\mathbb{F}}$

$$*$$

$$\begin{tikzcd} & 1 \arrow[dl, "id_n"'] \arrow[dr, "id_n"] & \\ \mathbf{Kl}^{T}_{\mathbb{F}}(n,n) \arrow[rr, "(\mathbf{Kl}^{\alpha}_{\mathbb{F}})_{n,n}"'] & & \mathbf{Kl}^{T'}_{\mathbb{F}}(n,n) \end{tikzcd}$$

$$n \xrightarrow{\eta} Tn \qquad \mapsto \qquad \begin{tikzcd} & n \arrow[dl, "\eta_n"'] \arrow[dr, "\eta'_n"] & \\ Tn \arrow[rr, "\alpha_n"'] & & T'n \end{tikzcd}$$

Proof: functoriality of $\mathbf{Kl}^{\alpha}_{\mathbb{F}} \colon \mathbf{Kl}^{T}_{\mathbb{F}} \to \mathbf{Kl}^{T'}_{\mathbb{F}}$

$$
\begin{array}{c}
m \xrightarrow{f} Tn \\
n \xrightarrow{g} Tp
\end{array}
\quad \mapsto \quad
\begin{array}{c}
m \xrightarrow{f} Tn \xrightarrow{\alpha_n} T'n \\
n \xrightarrow{g} Tp \xrightarrow{\alpha_p} T'p
\end{array}
$$

$$
\begin{tikzcd}
\mathbf{Kl}^{T}_{\mathbb{F}}(m,n) \times \mathbf{Kl}^{T}_{\mathbb{F}}(n,p) \arrow[r, "(\mathbf{Kl}^{\alpha}_{\mathbb{F}})_{m,n} \times (\mathbf{Kl}^{\alpha}_{\mathbb{F}})_{n,p}"] \arrow[d, "\circ"'] & \mathbf{Kl}^{T'}_{\mathbb{F}}(m,n) \times \mathbf{Kl}^{T'}_{\mathbb{F}}(n,p) \arrow[d, "\circ'"] \\
\mathbf{Kl}^{T}_{\mathbb{F}}(m,p) \arrow[r, "(\mathbf{Kl}^{\alpha}_{\mathbb{F}})_{m,p}"'] & \mathbf{Kl}^{T'}_{\mathbb{F}}(m,p)
\end{tikzcd}
$$

$$\downarrow \qquad\qquad \downarrow$$

$$
m \xrightarrow{f} Tn \xrightarrow{Tg} TTp \xrightarrow{\mu} Tp
\quad \mapsto \quad
\begin{tikzcd}
m \arrow[r, "f"] \arrow[d, "f"'] & Tn \arrow[r, "\alpha_n"] \arrow[dl, equal] & T'n \arrow[d, "T'g"] \\
Tn \arrow[d, "Tg"'] & & T'Tp \arrow[d, "T'\alpha_p"] \\
TTp \arrow[urr, "\alpha_{Tp}"] \arrow[rr, "(\alpha \circ_0 \alpha)_p"'] \arrow[d, "\mu"'] & & T'T'p \arrow[d, "\mu'_p"] \\
Tp \arrow[rr, "\alpha_p"'] & & T'p
\end{tikzcd}
$$

Proof functoriality of $\mathbf{Kl}_{\mathbb{F}}$

- We saw: each $\mathbf{Kl}_{\mathbb{F}}^{\alpha} \colon \mathbf{Kl}_{\mathbb{F}}^{T} \to \mathbf{Kl}_{\mathbb{F}}^{T'}$ is a functor.
- Now is $\mathbf{Kl}_{\mathbb{F}} \colon \mathbf{Mnd}(\mathbf{Set}) \to \mathbf{Law}$ functorial?
- I.e., given

$$T \xrightarrow{\alpha} T' \xrightarrow{\beta} T'',$$

does the following commute?

$$\begin{tikzcd}
\mathbf{Kl}_{\mathbb{F}}^{T} \arrow[r, "\mathbf{Kl}_{\mathbb{F}}^{\alpha}"] \arrow[dr, "\mathbf{Kl}_{\mathbb{F}}^{\beta\circ\alpha}"'] & \mathbf{Kl}_{\mathbb{F}}^{T'} \arrow[d, "\mathbf{Kl}_{\mathbb{F}}^{\beta}"] \\
& \mathbf{Kl}_{\mathbb{F}}^{T''}
\end{tikzcd}$$

Proof functoriality of $\mathbf{Kl}_{\mathbb{F}}$

Easy:

$$
\begin{tikzcd}
n \arrow[r, "f"] & Tp & \longmapsto & n \arrow[r, "f"] & Tp \arrow[r, "\alpha_p"] & T'p
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Kl}_{\mathbb{F}}^{T} \arrow[r, "\mathbf{Kl}_{\mathbb{F}}^{\alpha}"] \arrow[dr, "\mathbf{Kl}_{\mathbb{F}}^{\beta\circ\alpha}"'] & \mathbf{Kl}_{\mathbb{F}}^{T'} \arrow[d, "\mathbf{Kl}_{\mathbb{F}}^{\beta}"] \\
& \mathbf{Kl}_{\mathbb{F}}^{T''}
\end{tikzcd}
$$

$$
\begin{tikzcd}
n \arrow[r, "f"] & Tp \arrow[r, "\alpha_p"] \arrow[dr, "(\beta\circ\alpha)_p"'] & T'p \arrow[d, "\beta_p"] \\
& & T''p
\end{tikzcd}
$$

Functorial action $\mathbf{Law} \to \mathbf{Mnd}(\mathbf{Set})$

Let $F \colon \mathbb{K} \to \mathbb{L}$ in $\mathbf{Law}$.

- Induced natural transformation

$$\begin{gathered} \mathbf{T}_{\mathbb{K}}(X) \to \mathbf{T}_{\mathbb{L}}(X) \\ \int^{n} \mathbb{K}(n,1) \times X^{n} \to \int^{n} \mathbb{L}(n,1) \times X^{n} \\ [n,h,v] \mapsto [n,F(h),v]. \end{gathered}$$

- Naturality: excercise!

Naturality

$$
\begin{tikzcd}
X \xleftarrow{v} n \xrightarrow{h} 1 \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & X \xleftarrow{v} n \xrightarrow{Fh} 1 \arrow[ddd, mapsto] \\
& \int^n \mathbb{K}(n,1) \times X^n \arrow[r, "\int^n F_{n,1} \times X^n"] \arrow[d, "\int^n \mathbb{K}(n,1) \times f^n"'] & \int^n \mathbb{L}(n,1) \times X^n \arrow[d, "\int^n \mathbb{L}(n,1) \times f^n"] & \\
& \int^n \mathbb{K}(n,1) \times Y^n \arrow[r, "\int^n F_{n,1} \times Y^n"'] & \int^n \mathbb{L}(n,1) \times Y^n & \\
Y \xleftarrow{f} X \xleftarrow{v} n \xrightarrow{h} 1 \arrow[rrr, mapsto] & & & Y \xleftarrow{f} X \xleftarrow{v} n \xrightarrow{Fh} 1
\end{tikzcd}
$$

Functoriality of $T \colon \mathbf{Law} \to \mathbf{Mnd}(\mathbf{Set})$

Given $\mathbb{H} \xrightarrow{F} \mathbb{K} \xrightarrow{G} \mathbb{L}$, does the following commute?

$$
\begin{tikzcd}
\mathbf{T}_{\mathbb{H}} \arrow[r, "\mathbf{T}_F"] \arrow[dr, "\mathbf{T}_{GF}"'] & \mathbf{T}_{\mathbb{K}} \arrow[d, "\mathbf{T}_G"] \\
& \mathbf{T}_{\mathbb{L}}
\end{tikzcd}
$$

Functoriality of T: $\mathbf{Law} \to \mathbf{Mnd}(\mathbf{Set})$

Given $\mathbb{H} \xrightarrow{F} \mathbb{K} \xrightarrow{G} \mathbb{L}$, does the following commute?

$$
\begin{tikzcd}
X \xleftarrow{\sigma} n \xrightarrow{h} 1 \arrow[rrr, mapsto] \arrow[dddrrr, mapsto, bend right] & & & X \xleftarrow{\sigma} n \xrightarrow{Fh} 1 \arrow[ddd, mapsto] \\
& \mathbf{T}_{\mathbb{H}}(X) \arrow[r, "(\mathbf{T}_F)_X"] \arrow[dr, "(\mathbf{T}_{GF})_X"'] & \mathbf{T}_{\mathbb{K}}(X) \arrow[d, "(\mathbf{T}_G)_X"] & \\
& & \mathbf{T}_{\mathbb{L}}(X) & \\
& & & X \xleftarrow{\sigma} n \xrightarrow{GFh} 1
\end{tikzcd}
$$

Summary

We have functors:

$$
\begin{tikzcd}
\mathbf{Mnd}(\mathbf{Set}) \arrow[rr, bend left, "\mathbf{Kl}_{\mathbb{F}}"] & & \mathbf{Law}. \arrow[ll, bend left, "\mathbf{T}"]
\end{tikzcd}
$$

Do they form an equivalence?
No. Can anyone guess why?

The working side

Lemma

For any $\mathbb{L} \in \mathbf{Law}$ and $n \in \mathbb{N}$, the map

$$\mathbb{L}(n,1) \to \mathbf{T}_{\mathbb{L}}(n)$$

$$(n \xrightarrow{h} 1) \mapsto (\ n \xleftarrow{id_n} n \xrightarrow{h} 1\ )$$

is bijective.

The working side

Proof.

Surjectivity.

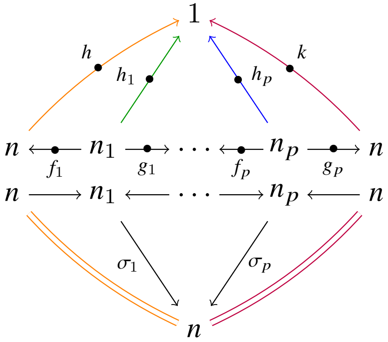
$$
\begin{tikzcd}
 & 1 & \\
p \arrow[ur, "h"] & & n \arrow[ul, dashed, "h\sigma"'] \arrow[ll, dashed, "\sigma"'] \\
p \arrow[rr, dashed, "\sigma"] \arrow[dr, "\sigma"'] & & n \\
 & n \arrow[ur, dashed, equal] &
\end{tikzcd}
$$

□

The working side

Proof.

Injectivity. Not so easy!



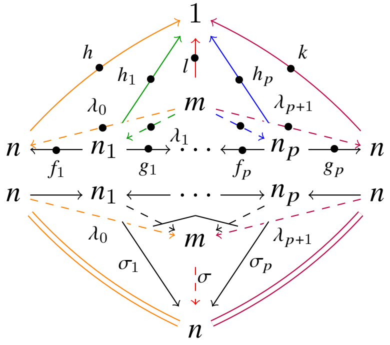
Consider any $(h, id_n) \sim (k, id_n)$.
Need $h = k$.

□

The working side

Proof.

Finite zig-zag $\rightsquigarrow$ take colimit in $\mathbf{Set}$: in $\mathbb{F}$.



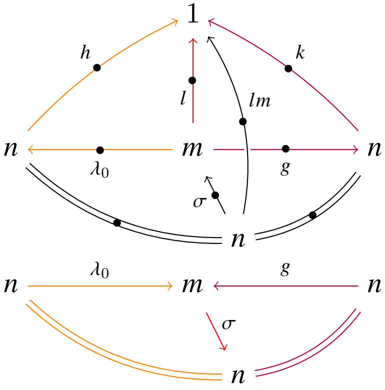
$$
\begin{aligned}
l := h\lambda_0 &= h(f_1\lambda_1) \\
&= h_1\lambda_1 \\
&\dots \\
&= h_p\lambda_p \\
&= (kg_p)\lambda_p \\
&= k\lambda_{p+1}.
\end{aligned}
$$

□

The working side

Proof.

Injectivity.



so $h = k$ as claimed.

□

The working side

We have proved $\mathbb{L}(n,1) \to \mathbf{T}_{\mathbb{L}}(n)$ bijective.

Corollary

$\mathbb{L} \cong \mathbf{Kl}_{\mathbb{F}}^{\mathbf{T}_{\mathbb{L}}}$.

Proof.

$$\begin{aligned} \mathbb{L}(n,p) &\cong \mathbb{L}(n,1)^p \\ &\cong \mathbf{T}_{\mathbb{L}}(n)^p && \text{(by the lemma)} \\ &= \mathbf{Kl}_{\mathbb{F}}^{\mathbf{T}_{\mathbb{L}}}(p,n). && \square \end{aligned}$$

The working side

We have proved $\mathbb{L}(n,1) \to \mathbf{T}_{\mathbb{L}}(n)$ bijective.

Corollary

$\mathbf{T} \colon \mathbf{Law} \to \mathbf{Mnd}(\mathbf{Set})$ *is fully faithful.*

Proof sketch.

- Consider any $\alpha \colon \mathbf{T}_{\mathbb{K}} \to \mathbf{T}_{\mathbb{L}}$ in $\mathbf{Mnd}(\mathbf{Set})$.
- Let $\overline{\alpha}_n$ denote:

$$\mathbb{K}(n,1) \xrightarrow{\cong} \mathbf{T}_{\mathbb{K}}(n) \xrightarrow{\alpha_n} \mathbf{T}_{\mathbb{L}}(n) \xrightarrow{\cong} \mathbb{L}(n,1)$$

- Then define $\alpha'_{p,q}$ by

$$\mathbb{K}(n,p) \xrightarrow{\cong} \mathbb{K}(n,1)^p \xrightarrow{\cong} \mathbf{T}_{\mathbb{K}}(n)^p \xrightarrow{\alpha_n^p} \mathbf{T}_{\mathbb{L}}(n)^p \xrightarrow{\cong} \mathbb{L}(n,1)^p \xrightarrow{\cong} \mathbb{L}(n,p)$$

- Show this defines a unique antecedent of α. □

The glitch

- Start from $F(X) = X^{\mathbb{N}}$.
- Consider $T = F^*$: $T(X) =$ set of finite-depth, $\mathbb{N}$-branching trees with leaves in X.
- $\mathbf{Kl}^T_{\mathbb{F}}(n, 1)$: same, with leaves in n.
- $\mathbf{T}_{\mathbf{Kl}^T_{\mathbb{F}}}(X) = \int^n T(n) \times X^n \hookrightarrow T(X)$:

$\boxed{\text{trees with finitely many distinct leaves!}}$

The glitch

T is not finitary.

Definition

A functor $F \colon \mathbf{Set} \to \mathbf{Set}$ is *finitary* if any $1 \to T(X)$ factors as

$$1 \xrightarrow{\xi} T(n) \xrightarrow{T(\sigma)} T(X).$$

(Counter?)example

Powerset functor $\mathcal{P}$.

The theorem

Definition

Let $\mathbf{Mnd}_f(\mathbf{Set}) \hookrightarrow \mathbf{Mnd}(\mathbf{Set})$ denote the full subcategory on finitary monads.

Recall:

$$
\begin{array}{rcl}
\mathbf{Law} & \to & \mathbf{CAT}/\mathbf{Set} \\
\mathbb{L} & \mapsto & \left(\begin{array}{ccc} \mathbf{Mod}(\mathbb{L}) & \xrightarrow{\mathbf{U}^{\mathbb{L}}} & \mathbf{Set} \\ M & \mapsto & M(1) \end{array} \right) \\
\mathbf{Mnd}(\mathbf{Set}) & \to & \mathbf{CAT}/\mathbf{Set} \\
T & \mapsto & \left(\begin{array}{ccc} T\text{-}\mathbf{Alg} & \xrightarrow{\mathbf{U}^{T}} & \mathbf{Set} \\ (TA \xrightarrow{\rho} A) & \mapsto & A \end{array} \right)
\end{array}
$$

The theorem

Theorem

We have

$$\begin{tikzcd}
\mathbf{Law} \arrow[rr, "\simeq"] \arrow[dr] & \arrow[d, phantom, "\simeq"] & \mathbf{Mnd}_f(\mathbf{Set}) \arrow[dl] \\
& \mathbf{CAT}/\mathbf{Set} &
\end{tikzcd}$$

Only part not covered yet: semantics preservation.

A pullback in $\mathbf{CAT}$

Let $\mathbf{i}_{\mathbb{F}}^*(X)(n) = \mathbf{Set}(n, X) = X^n$.

$$\begin{tikzcd}
\mathbf{P}_{\mathbb{L}} \arrow[r, dashed, "\mathbf{q}_{\mathbb{L}}"] \arrow[d, dashed, "\mathbf{p}_{\mathbb{L}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbb{L}, \mathbf{Set}]} \arrow[d] \\
\mathbf{Set} \arrow[r, "\mathbf{i}_{\mathbb{F}}^*"'] & {[\mathbb{F}^{op}, \mathbf{Set}]}
\end{tikzcd}$$

Object of $\mathbf{P}_{\mathbb{L}}$: set X, together with

- $X^h : X^n \to X^p$ for all $h : n \multimap p$ in $\mathbb{L}$,
- functorially,
- agrees with restriction on $\mathbb{F}^{op}$.

Morphism $X \to Y$:

- map $f : X \to Y$ with

$$\begin{tikzcd}
X^n \arrow[r, "f^n"] \arrow[d, "X^h"'] & Y^n \arrow[d, "Y^h"] \\
X^p \arrow[r, "f^p"'] & Y^p
\end{tikzcd}$$

First step

Lemma

We have $\mathbf{P}_{\mathbb{L}} \simeq [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$ *over sets.*

Proof of $\mathbf{P}_{\mathbb{L}} \simeq [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$

Object: set X, together with

- $X^h \colon X^n \to X^p$ for all $h \colon n \to\!\!\!\!\bullet\; p$,
- functorially,
- agrees with restriction on $\mathbb{F}^{op}$.

Proof of $\mathbf{P}_{\mathbb{L}} \simeq [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$

Object: set X, together with

- $X^h \colon X^n \to X^p$ for all $h \colon n \multimap p$,
- functorially,
- agrees with restriction on $\mathbb{F}^{op}$.
- $\rightsquigarrow$ preserves projections, hence products.

Proof of $\mathbf{P}_{\mathbb{L}} \simeq [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$

Object: set X, together with

- $X^h \colon X^n \to X^p$ for all $h \colon n \to\!\!\bullet\!\!\to p$,
- functorially,
- agrees with restriction on $\mathbb{F}^{op}$.
- $\rightsquigarrow$ preserves projections, hence products.

> $\rightsquigarrow$ object = strict model of $\mathbb{L}$, i.e., $M(n) = M(1)^n$.

Proof of $\mathbf{P}_{\mathbb{L}} \simeq [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$

We obtain a functor $\mathbf{P}_{\mathbb{L}} \to [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$.

- Objects embed as strict models.
- Morphisms f yield transformations $\alpha_n := f^n \colon X^n \to Y^n$.

Now:

- Surjective on objects.
- Faithful.
- Full?

Proof of $\mathbf{P}_{\mathbb{L}} \simeq [\mathbb{L}, \mathbf{Set}]_{\mathrm{fp}}$: fullness

For any natural transformation $\alpha \colon M \to N$ between models, taking $h = \pi_i$, $i \in n$:

$$\begin{tikzcd} X^n \arrow[r, "\alpha_n"] \arrow[d, "\pi_i"'] & Y^n \arrow[d, "\pi_i"] \\ X \arrow[r, "\alpha_1"'] & Y \end{tikzcd}$$

Taking $f = \alpha_1$, we have $\alpha_n = f^n$.

Second step

Lemma

We have $\mathbf{P}_{\mathbb{L}} \simeq \mathbf{T}_{\mathbb{L}}\text{-}\mathbf{Alg}$ *over sets.*

Proof sketch of $\mathbf{P}_{\mathbb{L}} \simeq \mathbf{T}_{\mathbb{L}}\text{-}\mathbf{Alg}$

Given set X with suitable actions $X^h \colon X^n \to X^p$.
$\mathbf{T}_{\mathbb{L}}$-algebra structure:

- given any $[h, \sigma] \in \mathbf{T}_{\mathbb{L}}(X)$, as in

$$X \xleftarrow{\sigma} n \xrightarrow{h} 1,$$

- return $1 \xrightarrow{\ulcorner \sigma \urcorner} X^n \xrightarrow{X^h} X$.

Omitted: check of monad algebra laws.

Conclusion

We have shown

$$\mathbf{Mod}(\mathbb{L}) \simeq \mathbf{P}_{\mathbb{L}} \cong \mathbf{T}_{\mathbb{L}}\text{-}\mathbf{Alg}$$

over sets, as desired, hence:

Theorem

We have

$$\begin{tikzcd} \mathbf{Law} \arrow[rr, "\simeq"] \arrow[dr] & \arrow[d, phantom, "\simeq"] & \mathbf{Mnd}_f(\mathbf{Set}) \arrow[dl] \\ & \mathbf{CAT}/\mathbf{Set} & \end{tikzcd}$$

Outline

Outline

Part III

Technical bases

Outline

Outline

Part IV

Correspondence between monads and theories, take 2

Outline