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CARTESIAN BICATEGORIES I

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Introduction

In this paper we continue the programme (initiated by Bénabou [2] and Lawvere [12]) of developing the theory of bicategories as a calculus of modules. Here we investigate some basic examples: sets and relations, additive relations, ordered sets and ideals.

These examples share with bicategories of the form **$\mathscr{V}$-mod** ($\mathscr{V}$-categories and $\mathscr{V}$-profunctors, $\mathscr{V}$ a cocomplete symmetric monoidal category) the structure of a *tensor product*

$$\mathbb{B}\times\mathbb{B}\xrightarrow{\otimes}\mathbb{B}$$

which is a homomorphism of bicategories, and which is coherently associative, symmetric and with identity I. If $\mathscr{V}$ is Cartesian, then every object X in $\mathbb{B}$ comes equipped with *diagonal*

$$\Delta_X : X\to X\otimes X$$

and *terminal*

$$t_X : X\to I$$

arrows which satisfy some basic laws. This leads to the first main notion of the paper, 'Cartesian bicategory'. A locally posetal bicategory is Cartesian if it has a symmetric tensor product, every object is a cocommutative comonoid object, every arrow is a lax comonoid homomorphism and comultiplication and counit have right adjoints. Alternatively, a locally posetal bicategory is Cartesian if the sub-bicategory of arrows with right-adjoints has finite biproducts, each hom-category has finite products and the obvious induced tensor product on arrows is functorial (Theorem 1.6). We deal only with locally posetal bicategories even though there is no doubt that the general notion of Cartesian bicategory may be developed to cover the examples of sets and spans, and categories and profunctors.

After describing the first consequences of our definition, we investigate the second main notion of 'discrete object' in a Cartesian bicategory. Modulo a 'functional completeness' axiom, bicategories of relations are characterized by Cartesianness and discreteness of every object (Sections 2, 3), and these properties together with small bicoproducts, effectiveness and generators characterize bicategories of relations of a Grothendieck topos (Section 6). Bicategories of ordered objects in an exact category can be characterized as follows: they are Cartesian, closed under the Kleisli construction, and the subbicategory of discrete objects is functionally complete and generates (in a suitable sense). That our notion of discrete object is correct for recovering the surrounding notion of 'set' is further supported by the following example. In the bicategory **SL** of sup-lattices considered in [10], our notion of discrete object coincides with the notion of 'discrete space' given there.

To finish, let us remark that our theory of relations differs from others in the literature (for example [6, 8]) in that *local limits and involution are not primitive.* As a gift for this more bicategorical setting, we have a theory flexible enough not just to cover the examples of relations and ideals, but also to give a simple and self-dual characterization of bicategories $\mathbb{B}$ of additive relations, as follows: $\mathbb{B}$ is Cartesian and cocartesian, every object is discrete and codiscrete, and reflexive and coreflexive arrows have splittings (Section 5).

1. Cartesian bicategories

In the following, $\mathbb{B}$ denotes a locally posetal bicategory. We usually denote objects of $\mathbb{B}$ by $X, Y, Z, \ldots$ and arrows by $r, s, t, \ldots$. Being locally posetal, $\mathbb{B}$ is in fact a 2-category.

Definition 1.1. A tensor product in $\mathbb{B}$ is a homomorphism of bicategories

$$\otimes : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

equipped with an identity object I and natural isomorphisms

$$\varrho : X \to X \otimes I; \qquad \gamma : X \otimes Y \to Y \otimes X,$$

$$\alpha : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

satisfying the classical coherence conditions (sufficient since $\mathbb{B}$ is locally posetal).

Definition 1.2. A Cartesian structure on a bicategory $\mathbb{B}$ consists of

(i) a tensor product in $\mathbb{B}$,

(ii) on every object X in $\mathbb{B}$, a comonoid structure. That is, arrows

$$\Delta_X : X \to X \otimes X, \qquad t_X : X \to I.$$

These data are required to satisfy the following axioms:

(Δ) The arrows Δ_X, t_X satisfy the equations for X to be a cocommutative comonoid object;

(U) Each arrow $r:X \to Y$ is a lax comonoid homomorphism. That is,

$$\Delta_Y \cdot r \subseteq (r \otimes r) \cdot \Delta_X \quad \text{and} \quad t_Y \cdot r \subseteq t_X;$$

(M) Comultiplication Δ_X and counit t_X have right adjoints Δ_X^*, t_X^*. The only cocommutative comonoid structure on X, with structure arrows having right adjoints, is (X, Δ_X, t_X).

In fact we will prove in Theorem 1.6 that a bicategory $\mathbb{B}$ admits at most one (up to iso) Cartesian structure, so justifying the name Cartesian bicategory.

Remark 1.3. (i) The arrows

$$\Delta_X^* : X \otimes X \to X, \qquad t_X^* : I \to X$$

which are right adjoints to Δ_X and t_X respectively, as stipulated in axiom (M), provide each object X with a commutative monoid structure which also satisfies axiom (U). Just observe that if f and g have right adjoints f^* and g^*, then $f \otimes g$ has a right adjoint $f^* \otimes g^*$ and, for any arrow r, we have $f \cdot r \subseteq g$ iff $r \cdot g^* \subseteq f^*$. In fact, $\mathbb{B}^{\mathrm{op}}$ (arrows reversed) has a Cartesian structure induced from $\mathbb{B}$.

(ii) Further consequences of axiom (M) are:

$$X \otimes X \xrightarrow{\Delta_X \otimes \Delta_X} X \otimes X \otimes X \otimes X \xrightarrow{1_X \otimes \gamma \otimes 1_X} X \otimes X \otimes X \otimes X$$

$$= X \otimes X \xrightarrow{\Delta_{X \otimes X}} X \otimes X \otimes X \otimes X \quad \text{(forgetting associativities);}$$

$$X \otimes X \xrightarrow{t_X \otimes t_X} I \otimes I \xrightarrow{\varrho} I = X \otimes X \xrightarrow{t_{X \otimes X}} I;$$

$$I \xrightarrow{\varrho} I \otimes I = I \xrightarrow{\Delta} I \otimes I, \qquad I \xrightarrow{1} I = I \xrightarrow{t} I.$$

To see this, notice that using the coherence conditions the left-hand arrows yield commutative comonoid structures on $X \otimes X$ and I.

(iii) The following is an (obvious generalization of a) result of Fox [7]. Consider a bicategory $\mathbb{B}$ with a tensor product. Then the tensor product is the biproduct iff every object has a cocommutative comonoid structure and every arrow is a comonoid homomorphism. Fox shows that if $\mathbb{B}$ is a symmetric monoidal bicategory, then the bicategory **HA**($\mathbb{B}$) of cocommutative comonoids in $\mathbb{B}$ (with comonoid homomorphisms) has biproducts, and the tensor product in $\mathbb{B}$ is the biproduct iff the forgetful functor **HA**($\mathbb{B}$) $\to \mathbb{B}$ is an isomorphism. The result extends to our notion of Cartesian bicategory. Given a bicategory $\mathbb{B}$ with a symmetric tensor product, let **HA**($\mathbb{B}$) be the bicategory of cocommutative comonoids in $\mathbb{B}$ and lax homomorphisms, and let $\mathbf{HA}_{\mathrm{R}}$($\mathbb{B}$) be the full subbicategory of **HA**($\mathbb{B}$) determined by the comonoids for which multiplication and counit have right adjoints. Then $\mathbf{HA}_{\mathrm{R}}(\mathbb{B})$ has a Cartesian structure. Moreover, $\mathbb{B}$ has a Cartesian structure iff $\mathbf{HA}_{\mathrm{R}}(\mathbb{B})\to\mathbb{B}$ is an isomorphism.

(iv) A tensor product on $\mathbb{B}$ induces in an obvious way a tensor product on $\mathbb{B}^{\mathrm{co}}$ (2-cells reversed). We call a Cartesian structure on $\mathbb{B}^{\mathrm{co}}$ a 'cocartesian' structure on $\mathbb{B}$.

Example 1.4. (i) $\mathbb{B}=\mathbf{Rel}(\mathscr{E})$, the bicategory of relations in a regular category $\mathscr{E}$ with a choice of products.

(ii) $\mathbb{B}=\mathbf{Ord}(\mathscr{E})$, the bicategory whose objects are ordered objects in a regular category $\mathscr{E}$ with a choice of products, and whose arrows are ideals.

(iii) $\mathbb{B}=$ the bicategory of inf-semilattices and (left-exact) functors.

(iv) $\mathbb{B}=\mathbf{Par}(\mathscr{E})$, the bicategory of partial maps in a left-exact category $\mathscr{E}$ with a choice of products satisfying all the axioms of a Cartesian bicategory except for the requirement for t_X to have a right adjoint. A stronger property than axiom (U) is satisfied in this case: every arrow is a *strict* comultiplication homomorphism.

Definition 1.5. An arrow $r:X\to Y$ in a bicategory $\mathbb{B}$ is called a *map* if it has a right adjoint r^*. Denote by $\mathbf{Map}(\mathbb{B})$ the subbicategory of $\mathbb{B}$ determined by these maps. Observe that a tensor product on $\mathbb{B}$ induces a tensor product on $\mathbf{Map}(\mathbb{B})$.

Theorem 1.6. *Let $\mathbb{B}$ be a locally posetal bicategory. If $\mathbb{B}$ has a Cartesian structure, then*

(i) $\mathbf{Map}(\mathbb{B})$ *has finite biproducts,*

(ii) $\mathbb{B}$ *locally has finite products and the identity arrow of I is the local terminal in $\mathbb{B}(I,I)$,*

(iii) *Biproducts and the biterminal object in* $\mathbf{Map}(\mathbb{B})$ *may be chosen so that the following formulas hold in* $\mathbb{B}$:

$$r\otimes s=(p^*\cdot r\cdot p)\cap(p^*\cdot s\cdot p)\quad (p\text{'s denote the appropriate projections}).$$

Conversely, if $\mathbb{B}$ satisfies properties (i) *and* (ii) *and the formulas of* (iii) *define a (functorial!) tensor product on $\mathbb{B}$, then $\mathbb{B}$ has a Cartesian structure.*

Proof. Suppose $\mathbb{B}$ has a Cartesian structure. To prove (i), by Remark 1.3(iii), we need just to show that every map is a comonoid homomorphism. From Remark 1.3(i), every arrow is also a monoid homomorphism. So, if f is a map, then $f^*\cdot\Delta^*\subseteq\Delta^*\cdot(f^*\otimes f^*)$ and $f^*\cdot t^*\subseteq t^*$. The opposite inclusion must hold for left adjoints and therefore, by axiom (U), $\Delta\cdot f=(f\otimes f)\cdot\Delta$ and $t\cdot f=t$.

To prove (ii), first notice that $\Delta^*\cdot(r\otimes s)\cdot\Delta\subseteq r$, since

$$\Delta^*\cdot(r\otimes s)\cdot\Delta\subseteq\Delta^*\cdot(r\otimes t^*\cdot t)\cdot\Delta=\Delta^*\cdot(1\otimes t^*)\cdot(r\otimes 1)\cdot(1\otimes t)\cdot\Delta$$
$$=\varrho\cdot(r\otimes 1)\cdot\varrho^{-1}=r,$$

(using the naturality of ϱ). Now, it is straightforward to show that the formula

$$r\cap s=\Delta^*\cdot(r\otimes s)\cdot\Delta$$

defines the local intersection. The local terminal is, of course, given by the formula

$$m_{X,Y} = t_Y^* \cdot t_X.$$

To prove (iii), it is just necessary to compute $(p^* \cdot r \cdot p) \cap (p^* \cdot r \cdot p)$ as $r \otimes s$.

Conversely, suppose $\mathbb{B}$ satisfies (i) and (ii) and we define a tensor on $\mathbb{B}$ by the formulae in (iii). It is straightforward that the conditions (Δ) and (M) for $\mathbb{B}$ to be Cartesian are satisfied. It remains to prove condition (U). Observe that composing on the left with a map, or on the right with the right adjoint of a map, preserves local intersections and local terminals. Using this fact and the definition of the tensor product it can be shown that $\Delta^* \cdot (r \otimes r) \cdot \Delta = r$ and condition (U) for Δ follows from the adjunction $\Delta \dashv \Delta^*$. Finally, condition (U) for t follows from the fact that t_X is the local terminal $m_{X,I}$. □

Corollary 1.7. *Let* $\mathbb{F} : \mathbb{B} \to \mathbb{D}$ *be a homomorphism, where* $\mathbb{B}, \mathbb{D}$ *are Cartesian bicategories. Then the following are equivalent*:

(i) $\mathbb{F}$ *is a strict monoidal functor*;

(ii) $\mathbb{F}$ *restricts to a strict monoidal functor* **Map**($\mathbb{B}$) → **Map**($\mathbb{D}$).

Proof. That (i) implies (ii) is obvious. If (ii) holds, then $\mathbb{F}$ preserves I, hence the terminal maps, and thus projections and diagonals. Thus $\mathbb{F}$ preserves local intersections and, by the previous theorem, the tensor product on arrows. □

2. Bicategories of relations

Definition 2.1. (i) An object X in a Cartesian bicategory is *discrete* when the multiplication Δ_X^* and the comultiplication Δ_X satisfy

(D) $$\Delta \cdot \Delta^* = (\Delta^* \otimes 1) \cdot (1 \otimes \Delta).$$

(We are forgetting the associativity in the middle.)

(ii) A Cartesian bicategory is called a '*bicategory of relations*' if every object is discrete.

Remark 2.2. If X, Y are discrete, then $X \otimes Y$ is also; I is always discrete. So, if $\mathbb{B}$ is a Cartesian bicategory, then the full subbicategory determined by the discrete objects is always a 'bicategory of relations'.

Example 2.3. (i) The bicategory of relations of a regular category $\mathscr{E}$ is, of course, a 'bicategory of relations'. To avoid confusion between these examples and the abstract notion we will always use quotation marks for the latter.

(ii) If $\mathbb{B}$ is **Ord**($\mathscr{E}$), then an object is discrete iff it is an equivalence relation. (See Section 6 on ordered objects.)

Theorem 2.4. *A 'bicategory of relations' admits transfer of variables; that is, it is* compact closed [11]. *In particular, the involution* $(\ )^\circ$ *is the identity on objects, and satisfies the following laws*:

$$(r\otimes 1)\cdot\Delta\subseteq(1\otimes r^\circ)\cdot\Delta\cdot r \quad \textit{and} \quad \Delta^*\cdot(r\otimes 1)\subseteq r\cdot\Delta^*(1\otimes r^\circ).$$

Proof. Define η_X and ε_X as follows:

$$\eta_X=I\xrightarrow{t_X^*}X\xrightarrow{\Delta_X}X\otimes X,\qquad \varepsilon_X=X\otimes X\xrightarrow{\Delta_X^*}X\xrightarrow{t_X}I.$$

We need to prove that $X\dashv X$; that is,

$$X\cong I\otimes X\xrightarrow{\eta\otimes 1}X\otimes X\otimes X\xrightarrow{1\otimes\varepsilon}X\otimes 1\cong X=1_X$$

and

$$X\cong X\otimes 1\xrightarrow{1\otimes\eta}X\otimes X\otimes X\xrightarrow{\varepsilon\otimes 1}I\otimes X\cong X=1_X.$$

The first composition can be computed as

$$\Delta^*\cdot(1\otimes t_X^*\cdot t_X)\cdot(1\otimes\Delta_X^*)\cdot(\Delta_X\otimes 1)\cdot(t_X^*\cdot t_X\otimes 1)\cdot\Delta=$$
$$=\Delta^*\cdot(1\otimes m_{X,X})\cdot\Delta_X\cdot\Delta_X^*\cdot(m_{X,X}\otimes 1)\cdot\Delta=1\cdot 1=1.$$

As for the second composition, observe that it can be deduced from the first by means of the symmetry isomorphism γ. So, $\mathbb{B}$ is compact closed and, as in [11], we can deduce that there is a natural isomorphism of ordered sets (transfer of variables)

$$\frac{X\otimes Y\xrightarrow{r}Z}{X\xrightarrow{\hat r}Z\otimes Y}\qquad (*)$$

given by $\hat r=X\cong X\otimes I\xrightarrow{1\otimes\eta}X\otimes Y\otimes Y\xrightarrow{r\otimes 1}Z\otimes Y$. The inverse is defined in a similar way by means of ε. The naturality of the correspondence (*) in X and Z means

$$\text{(X)}\qquad \frac{X'\otimes Y\xrightarrow{s\otimes 1}X\otimes Y\xrightarrow{r}Z}{X'\xrightarrow{s}X\xrightarrow{\hat r}Z\otimes Y},$$

$$\text{(Z)}\qquad \frac{X\otimes Y\xrightarrow{r}Z\xrightarrow{s}Z'}{X\xrightarrow{\hat r}Z\otimes Y\xrightarrow{s\otimes 1}Z'\otimes Y}.$$

Moreover, by defining the opposite of an arrow $X\xrightarrow{r}Y$ as

$$Y\xrightarrow{r^\circ}X=Y\cong Y\otimes I\xrightarrow{1\otimes\eta}Y\otimes X\otimes X\xrightarrow{1\otimes r\otimes 1}Y\otimes Y\otimes X\xrightarrow{\varepsilon\otimes 1}I\otimes X\cong Y,$$

one can prove that $(\ )^\circ$ becomes an involution on $\mathbb{B}$. That is, we have the properties

$$1^\circ=1,\quad (r\cdot s)^\circ=s^\circ\cdot r^\circ,\quad (r^\circ)^\circ=r\quad\text{and}\quad r\subseteq s\text{ implies }r^\circ\subseteq s^\circ.$$

Further, we have the naturality in Y of the correspondence $(*)$,

$$\text{(Y)} \qquad \frac{X\otimes Y' \xrightarrow{1\otimes s} X\otimes Y \xrightarrow{r} Z}{X \xrightarrow{\hat{r}} Z\otimes Y \xrightarrow{1\otimes s^\circ} Z\otimes Y'}\,.$$

Finally, to prove the stated laws, let us compute $\Delta^*\cdot(r\otimes r)\hat{}$ and $(r\cdot\Delta^*)\hat{}$:

$$\Delta^*\cdot(r\otimes r)\hat{} = (\Delta^*\cdot(1\otimes r)\cdot(r\otimes 1))\hat{} = (\Delta^*\cdot(1\otimes r))\hat{}\cdot r = (1\otimes r^\circ)\cdot\Delta^{*\hat{}}\cdot r$$

and

$$(r\cdot\Delta^*)\hat{} = (r\otimes 1)\cdot\Delta^{*\hat{}}.$$

Since $r\cdot\Delta^*\subseteq\Delta^*\cdot(r\otimes r)$, to prove the first law it suffices to show that $\Delta^{*\hat{}}=\Delta$. But

$$\Delta^{*\hat{}} = X \xrightarrow{\sim} X\otimes I \xrightarrow{1\otimes\eta} X\otimes X\otimes X \xrightarrow{\Delta^*\otimes 1} X\otimes X = \Delta\cdot\Delta^*\cdot(1\otimes m_{X,X})\cdot\Delta = \Delta.$$

In a similar way we can prove the second of the two laws. □

Observe that axiom (U) does not play any role in the proof that $\mathbb{B}$ is compact closed. Axiom (U) is needed only to prove the two stated laws.

A consequence of the discreteness axiom is that the *order between maps is discrete*; that is, **Map**($\mathbb{B}$) is a category.

Lemma 2.5. *In a 'bicategory of relations' an arrow $X \xrightarrow{r} Y$ is a map iff it is a comonoid homomorphism iff $r \dashv r^\circ$.*

Proof. We have already proved that a map is a comonoid homomorphism (Theorem 1.6). We need just to prove that if an arrow r is a comonoid homomorphism, then r° is right adjoint to r. By using the definition of η and ε, we can compute the compositions

$$\begin{aligned}
r\cdot r^\circ = {} & Y \xrightarrow{\sim} I\otimes Y \xrightarrow{t^*\otimes 1} X\otimes Y \xrightarrow{\Delta\otimes 1} X\otimes X\otimes Y \xrightarrow{1\otimes r\otimes 1} \\
& X\otimes Y\otimes Y \xrightarrow{1\otimes\Delta^*} X\otimes Y \xrightarrow{1\otimes t} X\otimes I \xrightarrow{\sim} X \xrightarrow{r} Y \\
= {} & Y \xrightarrow{\sim} I\otimes Y \xrightarrow{t^*\otimes 1} X\otimes Y \xrightarrow{\Delta\otimes 1} X\otimes X\otimes Y \xrightarrow{r\otimes r\otimes 1} \\
& Y\otimes Y\otimes Y \xrightarrow{1\otimes\Delta^*} Y\otimes Y \xrightarrow{1\otimes t} Y\otimes I \xrightarrow{\sim} Y \\
= {} & Y \xrightarrow{\sim} I\otimes Y \xrightarrow{t^*\otimes 1} X\otimes Y \xrightarrow{r\otimes 1} Y\otimes Y \xrightarrow{\Delta\otimes 1} \\
& Y\otimes Y\otimes Y \xrightarrow{1\otimes\Delta^*} Y\otimes Y \xrightarrow{1\otimes t} Y\otimes I \xrightarrow{\sim} Y \\
\subseteq {} & Y \xrightarrow{\sim} I\otimes Y \xrightarrow{t^*\otimes 1} Y\otimes Y \xrightarrow{\Delta^*} Y \xrightarrow{\Delta} Y\otimes Y \xrightarrow{1\otimes t} Y\otimes I \xrightarrow{\sim} Y \\
= {} & 1\cdot 1 = 1,
\end{aligned}$$

$$r^\circ \cdot r = X \xrightarrow{r} Y \xrightarrow{\sim} I \otimes Y \xrightarrow{t^* \otimes 1} X \otimes Y \xrightarrow{\Delta \otimes 1} X \otimes X \otimes Y \xrightarrow{1 \otimes r \otimes 1}$$

$$X \otimes Y \otimes Y \xrightarrow{1 \otimes \Delta^*} X \otimes Y \xrightarrow{1 \otimes t} X \otimes I \xrightarrow{\sim} X$$

$$= X \xrightarrow{\sim} I \otimes X \xrightarrow{t^* \otimes 1} X \otimes X \xrightarrow{\Delta \otimes 1} X \otimes X \otimes X \xrightarrow{1 \otimes r \otimes r}$$

$$X \otimes Y \otimes Y \xrightarrow{1 \otimes \Delta^*} X \otimes Y \xrightarrow{1 \otimes t} X \otimes I \xrightarrow{\sim} X$$

$$\supseteq X \xrightarrow{\sim} I \otimes X \xrightarrow{t^* \otimes 1} X \otimes X \xrightarrow{\Delta \otimes 1} X \otimes X \otimes X \xrightarrow{1 \otimes \Delta^*}$$

$$X \otimes X \xrightarrow{1 \otimes r} X \otimes Y \xrightarrow{1 \otimes t} X \otimes I \xrightarrow{\sim} X$$

$$= X \xrightarrow{\sim} I \otimes X \xrightarrow{t^* \otimes 1} X \otimes X \xrightarrow{\Delta \otimes 1} X \otimes X \otimes X \xrightarrow{1 \otimes \Delta^*}$$

$$X \otimes X \xrightarrow{1 \otimes t} X \otimes 1 \xrightarrow{\sim} X$$

$$\supseteq X \xrightarrow{\sim} I \otimes X \xrightarrow{t^* \otimes 1} X \otimes X \xrightarrow{\Delta^*} X \xrightarrow{\Delta} X \otimes X \xrightarrow{1 \otimes t} X \otimes I \xrightarrow{\sim} X$$

$$= 1 \cdot 1 = 1. \qquad \square$$

Observe that in the proof of $r^\circ \cdot r \supseteq 1$ the discreteness axiom is not used and that the role of the hypothesis that r is a comonoid homomorphism can be split in two: that r is a comultiplication homomorphism is used in the proof of $r \cdot r^\circ \subseteq 1$ (r is a 'partial map'); that r is a counit homomorphism is used in the proof of $r^\circ \cdot r \supseteq 1$ (r is 'entire' or 'everwhere defined').

Corollary 2.6. *In a 'bicategory of relations':*

(i) *if f is a map, then $f^* = f^\circ$: in particular $\Delta^* = \Delta^\circ$ and $t^* = t^\circ$;*

(ii) *if f, g are maps and $f \subseteq g$, then $f = g$;*

(iii) *r is a partial map iff r is a comultiplication homomorphism; composing on the right with a partial map r or on the left with r° preserves local intersections;*

(iv) *r is entire iff it is a counit homomorphism; composing on the right with an entire morphism r or on the left with r° preserves local terminals; if a composite $s \cdot r$ is entire, then r is entire.*

Proof. (i) Follows from uniqueness of the adjoints and the previous lemma.

(ii) Obvious since adjoints are opposites.

(ii) If r is a comultiplication homomorphism, then by Lemma 2.5 it follows that $r \cdot r^\circ \subseteq 1$, that is, that r is a partial map; conversely, if $r \cdot r^\circ \subseteq 1$, then by the law of Theorem 2.4 we have

$$\Delta \cdot r \supseteq (r \cdot r^\circ \otimes 1) \cdot \Delta \cdot r = (r \otimes 1) \cdot (r^\circ \otimes 1) \cdot \Delta \cdot r \supseteq (r \otimes 1) \cdot (1 \otimes r) \cdot \Delta$$
$$= (r \otimes r) \cdot \Delta.$$

The preservation property is now immediate from the definition of local intersections.

(iv) It has already been observed that if r is a counit homomorphism, then r is entire. Conversely, suppose r is entire. Then because t is a local maximum $t\cdot r\supseteq t\cdot r^\circ\cdot r\supseteq t$. So $m\cdot r=t^\circ\cdot t\cdot r=t^\circ\cdot t=m$. Finally, if $t\cdot s\cdot r=t$, then $t\cdot r\supseteq t\cdot s\cdot r=t$. □

In the presence of discreteness we can also improve Corollary 1.7 as follows:

Corollary 2.7. *Let $\mathbb{F}\cdot\mathbb{B}\to\mathbb{D}$ be a homomorphism of bicategories between 'bicategories of relations'. Then the following are equivalent:*

(i) *$\mathbb{F}$ is a strict monoidal homomorphism;*

(ii) *$\mathbb{F}$ restricts to a strict monoidal functor* **Map**($\mathbb{B}$)→**Map**($\mathbb{B}$);

(iii) *$\mathbb{F}$ preserves local intersections and I.* □

The main point which we have not so far covered is that (iii) implies (i). This reduces easily to showing, under assumption (iii), that projections are preserved (up to isos) which follows from

Lemma 2.8. *In a bicategory of relations let $p_X:X\otimes Y\to X$ and $p_Y:X\otimes Y\to Y$ be the projections. Then*

(i) $p_Y\cdot p_X^\circ=m_{X,Y}$, $p_X^\circ\cdot p_X\cap p_Y^\circ\cdot p_Y=1$;

(ii) *if f,g are maps such that $g\cdot f^\circ=m_{X,Y}$ and $f^\circ\cdot f\cap g^\circ\cdot g=1$, then the map $\langle f,g\rangle=(f\otimes g)\cdot\Delta$ is an isomorphism, and hence $f\cong p_X$ and $g\cong p_Y$.*

Proof. Properties (i) are easily proved for any Cartesian bicategory. To prove (ii) first observe that the condition $g^\circ\cdot g\cap f^\circ\cdot f=1$ means exactly that $\langle f,g\rangle^\circ\cdot\langle f,g\rangle=1$. Then

$$\begin{aligned}
t\cdot\langle f,g\rangle^\circ&=t\cdot\Delta^\circ\cdot(f^\circ\otimes g^\circ)=t\cdot g\cdot\Delta^\circ\cdot(1\otimes g^\circ)\cdot(f^\circ\otimes 1)\\
&\supseteq t\cdot\Delta^\circ\cdot(g\otimes 1)\cdot(f^\circ\otimes 1)=t\cdot\Delta^\circ\cdot(g\cdot f^\circ\otimes 1)\\
&=t\cdot\Delta^\circ\cdot(m\otimes 1)=t\cdot\Delta^\circ(t^\circ\cdot t\otimes 1)=t\cdot\Delta^\circ\cdot(t^\circ\otimes 1)\cdot(t\otimes 1)\\
&\supset\Delta^\circ\cdot(1\otimes t)\cdot(t\otimes 1)=\Delta^\circ\cdot(t\otimes t)=t.
\end{aligned}$$

Thus $\langle f,g\rangle^\circ$ is entire ($\langle f,g\rangle\cdot\langle f,g\rangle^\circ=1$). So $\langle f,g\rangle$ is an isomorphism. □

Remarks 2.9. (i) Every strict monoidal homomorphism $\mathbb{F}$ between Cartesian bicategories preserves discrete objects as well as η and ε. Therefore $\mathbb{F}$ preserves also the involution when restricted to the full subbicategory determined by the discrete objects.

(ii) Every bicategory of relations satisfies Freyd's modular law [8]: $s\cdot r\cap t\subseteq(t\cdot r^\circ\cap s)\cdot r$. To see this notice that

$$s\cdot r\cap t=\Delta^\circ\cdot(s\cdot r\otimes t)\Delta=\Delta^\circ\cdot(s\otimes t)\cdot(r\otimes 1)\cdot\Delta$$
$$\subseteq\Delta^\circ\cdot(s\otimes t)\cdot(1\otimes r^\circ)\cdot\Delta\cdot r=\Delta^\circ\cdot(s\otimes t\cdot r^\circ)\cdot\Delta\cdot r$$
$$=(s\cap t\cdot r^\circ)\cdot r.$$

(iii) Given a multisorted theory $\mathbb{T}$ formulated in the $(\exists,\wedge,t,=)$-fragment of logic, we can construct out of $\mathbb{T}$ a 'bicategory of relations' $\mathbb{B}(\mathbb{T})$ as follows. Objects $\boldsymbol{\alpha}$ are finite words $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$ of basic sorts; arrows $r:\boldsymbol{\alpha}\to\boldsymbol{\beta}$ are (equivalence classes of) formulae $r(\boldsymbol{x},\boldsymbol{y})$ having free variables of sorts $(\boldsymbol{\alpha},\boldsymbol{\beta})$; 2-cells are entailments. Composition is given by $s\cdot r(\boldsymbol{x},\boldsymbol{z})=\exists y(r(\boldsymbol{x},\boldsymbol{y})\wedge s(\boldsymbol{y},\boldsymbol{z}))$ and the tensor product by the conjunction. The comultiplication Δ is the formula

$$\Delta(x,x',x'')=((x=x')\wedge(x=x'')),$$

and the discreteness axiom expresses the symmetry of equality. The converse can also be proved: given a (small and locally small) 'bicategory of relations' $\mathbb{B}$, there exists a theory $\mathbb{T}$ such that $\mathbb{B}(\mathbb{T})$ is equivalent to $\mathbb{B}$. From this point of view, we can think of a small and locally small Cartesian bicategory $\mathbb{B}$ as a theory in which equality is not assumed symmetric.

(iv) Let **SL** be the bicategory of sup-lattices considered in [10]. Then **SL** has a tensor product. Let **Mon(SL)** be the bicategory of commutative monoids in **SL** with morphisms $f:X\to Y$ being the sup-lattice ones such that $f(x\cdot y)\le f(x)\cdot f(y)$ and $f(1)\le 1$. Let $\mathbb{B}$ be the full subbicategory of **Mon(SL)** determined by the monoids such that the multiplication and unit have left adjoints. Then, by Remark 1.3(i), $\mathbb{B}$ is Cartesian. Observe that $\mathbf{Mon(SL)}^{co}$ is the bicategory of cocomplete, symmetric monoidal categories and cocontinuous monoidal functors between them. A reason to choose such morphisms in **Mon(SL)**, besides the formal fact that with these morphisms it is Cartesian, is that if PX, PY are sup-lattices of 'parts', then a relation $r:X\to Y$ is the same as a cocontinuous functor $f:PY\to PX$, and the conditions $f(x\cdot y)\le f(x)\cdot f(y)$, $f(1)\le 1$ are satisfied by any functor since PX, PY are left exact.

The subbicategory $\mathbb{B}_{disc}$ of the bicategory $\mathbb{B}$ determined by the discrete objects is a 'bicategory of relations' (see Remark 2.2). We will show now that $\mathbf{Map}(\mathbb{B}_{disc})$ coincides with the category of discrete spaces defined in [10]. The objects of $\mathbf{Map}(\mathbb{B}_{disc})$ are clearly locales in which the external maps $X\xrightarrow{\Delta}X\otimes X$ and $X\xrightarrow{t}I$ are cocontinuous. We need to show that our discreteness axiom is equivalent to the openness of the pair $\Delta\dashv\Delta^*$, that is, $\Delta\cdot\Delta^*\cdot(\Delta^*\otimes 1)\supseteq\Delta^*\cdot(1\otimes\Delta)$. But

$$\Delta\cdot\Delta^*=\Delta\cdot\Delta^*\cdot(1\otimes\Delta^*)\cdot(1\otimes\Delta)=\Delta\cdot\Delta^*\cdot(\Delta^*\otimes 1)\cdot(1\otimes\Delta)$$
$$=\Delta^*\cdot(1\otimes\Delta)(1\otimes\Delta)=\Delta^*\cdot(1\otimes 1\otimes\Delta)\cdot(1\otimes\Delta)$$
$$=\Delta^*\cdot(1\otimes((1\otimes\Delta)\cdot\Delta))$$
$$=(\Delta^*\otimes\Delta^*)\cdot(1\otimes\gamma\otimes 1)\cdot(1\otimes((1\otimes\Delta)\cdot\Delta))$$
$$=(\Delta^*\otimes\Delta^*)\cdot(1\otimes\gamma\otimes 1)\cdot(1\otimes((\Delta\otimes 1)\cdot\Delta))$$
$$=(\Delta^*\otimes\Delta^*)(1\otimes\gamma\otimes 1)\cdot(1\otimes\Delta\otimes 1)(1\otimes\Delta)$$

$$= (\Delta^* \otimes \Delta^*)(1 \otimes \Delta \otimes 1)(1 \otimes \Delta) = (\Delta^* \otimes \Delta^*)(1 \otimes ((\Delta \otimes 1)\Delta))$$
$$= (\Delta^* \otimes \Delta^*) \cdot (1 \otimes (1 \otimes \Delta)\Delta) = (\Delta^* \otimes \Delta^*) \cdot (1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)$$
$$= (\Delta^* \otimes \Delta^* \cdot \Delta) \cdot (1 \otimes \Delta) = (\Delta^* \otimes 1)(1 \otimes \Delta).$$

In the other direction, if the discreteness axiom holds, then from the laws of Theorem 2.4 we have: $\Delta^*(1 \otimes \Delta) \subseteq \Delta \cdot \Delta^*(\Delta^\circ \otimes 1)$, and because $\Delta^* = \Delta^\circ$ (Corollary 2.6) we have $\Delta^* \cdot (1 \otimes \Delta) \subseteq \Delta \cdot \Delta^* \cdot (\Delta^* \otimes 1)$. Thus, since every arrow is a lax monoid homomorphism and since $\Delta \dashv \Delta^*$, we get the opposite inclusion also.

From Corollary 2.5 we have that an arrow f is a map iff the adjoint f° is a monoid homomorphism, so that **Map**($\mathbb{B}_{\text{disc}}$) is the same as the category of discrete spaces.

3. The characterization theorem

We now give a proof of the characterization theorem for bicategories of relations of regular categories. The proof here is slightly different from the one given by Freyd [8]. One difference is that by the systematic use of our calculus of $(\otimes, I, \Delta, t)$ we have been able to avoid the use of argument by contradiction.

Definition 3.1. A 'bicategory of relations' is *functionally complete* if for every arrow $r: X \to I$ there exists a map $i: X_r \to X$ such that $i^\circ \cdot i = 1$ and $t \cdot i^\circ = r$. The map i is called a *tabulation* of r [8].

Remark 3.2. The 'bicategory of relations' $\mathbb{B}(\mathbb{T})$ associated to a theory $\mathbb{T}$ (see Remark 2.9(ii)) is, in general, not functionally complete. To say that $\mathbb{B}(\mathbb{T})$ is functionally complete is to say that for each formula $r(x)$ of sort $\boldsymbol{\alpha}$ there exists a definable function symbol $i_r: \boldsymbol{\alpha}_r \to \boldsymbol{\alpha}$ such that $r(x)$ is equivalent to $\exists x'(i(x') = x)$ and such that $i(x) = i(x'') \models x = x''$. Given $\mathbb{T}$ we can of course add function symbols to have an extension $\mathbb{T}'$ of $\mathbb{T}$ (having the same models as $\mathbb{T}$) such that $\mathbb{T}'$ is functionally complete. We will see later on (Section 4) how to construct $\mathbb{B}(\mathbb{T}')$ out of $\mathbb{B}(\mathbb{T})$ in a purely algebraic way.

Lemma 3.3. *Let i be a tabulation of $r: X \to I$. Then*

(i) *for each map $f: Z \to X$ such that $t \cdot f^\circ \subseteq r$ there exists a unique map $h: Z \to X_r$ such that $f = i \cdot h$;*

(ii) *if $t \cdot f^\circ = r$, then $t \cdot h^\circ = t$; that is, h° is entire.*

Proof. (i) Define h as $h = i^\circ \cdot f$. Then h is a partial map, being the composite of partial maps. Further $t \cdot h = t \cdot i^\circ \cdot f = r \cdot f \supseteq t \cdot f^\circ \cdot f \supset t$, and thus $t \cdot h = t$ and h is entire. Therefore, h is a map and $i \cdot h = i \cdot i^\circ \cdot f \subseteq f$. Hence from Corollary 2.6, $i \cdot h = f$. Uniqueness follows from the fact that $i^\circ \cdot i = 1$.

(ii) $t \cdot h^\circ = t \cdot f^\circ \cdot i = r \cdot i = t$; that is, h° is entire. Observe that since h is a map h° entire means that $h \cdot h^\circ = 1$. □

Corollary 3.4. *Every arrow $r: X \to Y$ has a tabulation, that is, a jointly monic pair of maps f, g such that*

(i) $g \cdot f^\circ = r$;

(ii) *if x, y is another pair of maps such that $y \cdot x^\circ \subseteq r$, then there exists a unique map h such that $f \cdot h = x$ and $g \cdot h = y$ and such that, if $y \cdot x^\circ = r$, then h° is entire.*

Proof. (i) Consider the transpose $\hat{r}: X \otimes Y \to I = \varepsilon \cdot (r \otimes 1)$ of r arising from the compact closedness. Let $i = \langle f, g \rangle$ be a tabulation of $\hat{r}$. Then f, g are jointly monic. To prove that $g \cdot f^\circ = r$, let us compute the arrow corresponding to $t \cdot i^\circ = \hat{r}$ by the compact closedness:

$$X \xrightarrow{\sim} X \otimes I \xrightarrow{1 \otimes \eta} X \otimes Y \otimes Y \xrightarrow{\hat{r} \otimes 1} I \otimes Y \xrightarrow{\sim} Y =$$

$$= X \xrightarrow{\sim} X \otimes I \xrightarrow{1 \otimes \eta} X \otimes Y \otimes Y \xrightarrow{f^\circ \otimes g^\circ}$$

$$Z \otimes Y \otimes Y \xrightarrow{\Delta^\circ \otimes 1} Z \otimes Y \xrightarrow{t \otimes 1} I \otimes Y \xrightarrow{\sim} Y$$

$$= X \xrightarrow{f^\circ} Z \xrightarrow{\sim} Z \otimes I \xrightarrow{1 \otimes \eta} Z \otimes Y \otimes Y \xrightarrow{1 \otimes g^\circ \otimes 1}$$

$$Z \otimes Z \otimes Y \xrightarrow{\varepsilon \otimes 1} I \otimes Y \xrightarrow{\sim} Y$$

$$= X \xrightarrow{f^\circ} Z \xrightarrow{(g^\circ)^\circ} Y = X \xrightarrow{f^\circ} Z \xrightarrow{g} Y.$$

Thus $g \cdot f^\circ = r$.

(ii) We need just to prove that the condition $y \cdot x^\circ \subseteq r$ is equivalent to $t \cdot \langle x, y \rangle^\circ \subseteq \hat{r}$ and then apply Lemma 3.3. This follows by a calculation similar to the above using compact closedness. □

Theorem 3.5. *Let $\mathbb{B}$ be a functionally complete bicategory of relations. Then*

(i) *$\mathscr{E} =$* **Map**$(\mathbb{B})$ *is a regular category – that is, a left exact category in which extremal epis are stable under pullback and every arrow factors as an extremal epi followed by a mono;*

(ii) *the function assigning to each relation $\langle f, g \rangle$ of $\mathscr{E}$ the arrow $g \cdot f^\circ$ of $\mathbb{B}$ extends to a biequivalence of bicategories.*

Proof. (i) From Corollary 3.4, a pullback of r, s in $\mathscr{E} =$ **Map**$(\mathbb{B})$ is a tabulation of $s^\circ \cdot r$; so $\mathscr{E}$ is left exact, and monos i in $\mathscr{E}$ are characterized by the equation $i^\circ \cdot i = 1$. Now let $f: X \to Y$ be an arrow in $\mathscr{E}$ and consider $Y \xrightarrow{f^\circ} X \xrightarrow{t} I$. Lemma 3.3 provides a factorization of f as $f = i \cdot h$, with i mono and h° entire. From this it follows that a map h is extremal epi iff h° is entire; that is, iff $h \cdot h^\circ = 1$. Finally, if $f \cdot h' = h \cdot f'$ is a pullback in $\mathscr{E}$ and f is an extremal epi, then $t \cdot f'^{\circ} \supseteq t \cdot h' \cdot f'^{\circ} = t \cdot f^{\circ} \cdot h = t \cdot h = t$. Hence f'° is entire and f' is extremal epi. Thus $\mathscr{E}$ is a regular category.

(ii) Since the assignment is the identity on objects and locally an isomorphism we need only prove the functoriality. But this is an instance of Corollary 3.4(ii). □

Since monoidal homomorphisms between 'bicategories of relations' preserve tabulations, we have

Corollary 3.6. *Suppose* $\mathbb{B}$ *and* $\mathbb{D}$ *are functionally complete 'bicategories of relations'. Then the category of monoidal homomorphisms* $\mathbb{B} \to \mathbb{B}$ *is equivalent to the category of left exact extremal-epi preserving functors* $\mathbf{Map}(\mathbb{B}) \to \mathbf{Map}(\mathbb{D})$. □

Remark 3.7. Using the above characterization theorem it is easy to characterize bicategories of relations of other important classes of categories. For example, bicategories of relations of Heyting categories (those regular categories such that for each $X \xrightarrow{f} Y$ the inverse image functor $f^*: \mathbf{Sub}(Y) \to \mathbf{Sub}(X)$ has a right adjoint $\forall f$) can be characterized as the functionally complete bicategories of relations having all right Kan extensions (and thus all right liftings, from compact closedness). Again, bicategories of relations of geometric (coherent) categories (those regular categories having pullback-stable (finite) unions of subobjects) can be characterized as those functionally complete 'bicategories of relations' which are locally (finitely) cocomplete, with local unions preserved by composition (*distributive* 'bicategories of relations'). Finally, bicategories of relations of elementary toposes can be characterized as those functionally complete 'bicategories of relations' such that $\mathbb{B}(X, -)$ is representable in $\mathbf{Map}(\mathbb{B})$. A good set of operations and equations to express this representability has been found by Freyd.

We will investigate bicategories of relations of Grothendieck toposes in the last section.

4. Ordered objects and ideals

In this section we characterize the bicategory of ordered objects and ideals of an exact category $\mathscr{E}$. An ordered object in $\mathscr{E}$ is a relation $p: X \to X$ such that $1 \subseteq p$ and $p \cdot p \subseteq p$; that is, it is just a monad in $\mathbb{B} = \mathbf{Rel}(\mathscr{E})$. An equivalence relation in $\mathscr{E}$ is a symmetric monad in $\mathbb{B}$; that is, a monad such that $p = p^{\circ}$. Comonads in $\mathbb{B}$ are also important; we show now, following [8], that the bicategory of symmetric comonads provides the free functional completion of a 'bicategory of relations'. We begin with

Lemma 4.1. *In a 'bicategory of relations':*

(i) *the class of symmetric comonads coincides with the class of 'coreflexives' (arrows* a *such that* $a \subseteq 1$);

(ii) *if a and b are two reflexives on X, then* $b\cdot a=a\cap b$.

Proof.

$$a=1\cap a=\Delta^\circ\cdot(1\otimes a)\cdot\Delta\subset\Delta^\circ\cdot(a^\circ\otimes 1)\cdot\Delta\cdot a\subseteq 1\cap a^\circ=a^\circ.$$

If $b\subset 1$, then $b\cdot a\subseteq a\cap b$. But

$$a\cap b=\Delta^\circ\cdot(a\otimes b)\Delta=\Delta^\circ\cdot(1\otimes b)\cdot(a\otimes 1)\Delta$$
$$\subseteq\Delta^\circ(1\otimes b)(1\otimes a^\circ)\Delta a=(1\cap b\cdot a^\circ)\cdot a\subset b\cdot a. \qquad\square$$

As in any bicategory, a Kleisli object for a monad p is an object X_p and an arrow $X\to X_p$ which represent the functor p-Alg$(X,-)$. The representability implies that the arrow $e:X\to X_p$ has a right adjoint e^* such that $e^*\cdot e=p$ and (in the locally posetal case) $e\cdot e^*=1$. These two equations characterize the Kleisli construction for p as a splitting of the idempotent p. Dually for a comonad a, the splitting $i^*\cdot i=1$ and $i\cdot i^*=a$ of a as an idempotent characterizes the Kleisli construction of the comonad a. If $\mathbb{B}$ is a 'bicategory of relations', then the comonad a is symmetric and the adjoint i^* of the splitting i of the comonad a coincides with i°. Observe that 'bicategories of relations' can admit a Kleisli construction only for symmetric monads and comonads.

Lemma 4.2. *A bicategory of relations is functionally complete iff every symmetric comonad has a Kleisli construction; that is, iff coreflexives split.*

Proof. First observe that the function 'domain',

$$\mathbb{B}(X,I)\xrightarrow{\mathscr{D}}\mathrm{Cor}(X)=\{a\in\mathbb{B}(X,X)\,|\,a\subset 1\}$$

which associates to each $r:X\to I$ its domain $\mathscr{D}(r)=1\cap r^\circ\cdot r$, is an isomorphism, whose inverse is given by composition with t. For, if a is a coreflexive, then

$$\mathscr{D}(t\cdot a)=1\cap a^\circ\cdot t^\circ\cdot t\cdot a\supseteq 1\cap a^\circ\cdot a=a,$$

and

$$\mathscr{D}(t\cdot a)=\Delta^\circ\cdot(1\otimes a^\circ\cdot t^\circ\cdot t\cdot a)\cdot\Delta=\Delta^\circ\cdot(1\otimes a^\circ)\cdot(1\otimes t^\circ\cdot t\cdot a)\Delta$$
$$\subseteq a^\circ\cdot\Delta^\circ\cdot(a\otimes 1)\cdot(1\otimes t^\circ\cdot t\cdot a)\Delta=a^\circ\cdot\Delta^\circ\cdot(a\otimes t^\circ\cdot t\cdot a)\cdot\Delta$$
$$=a\cdot(a\cap t^\circ\cdot t\cdot a)\subset a.$$

Conversely, if $r:X\to I$, then $t\cdot\mathscr{D}(r)=t\cdot(1\cap r^\circ\cdot r)\subseteq t\cdot r^\circ\cdot r\subseteq r$, because $t_I=1_I$. Further,

$$t\cdot(1\cap r^\circ\cdot r)\supseteq r\cdot(1\cap r^\circ\cdot r)=r\cdot\Delta^\circ\cdot(1\otimes r^\circ\cdot r)\Delta$$
$$=r\cdot\Delta^\circ\cdot(1\otimes r^\circ)\cdot(1\otimes r)\cdot\Delta$$
$$\supseteq\Delta^\circ\cdot(r\otimes 1)\cdot(1\otimes r)\cdot\Delta=\Delta^\circ\cdot(r\otimes r)\cdot\Delta=r.$$

Now, suppose $\mathbb{B}$ is functionally complete and let a be coreflexive. If i is a tabulation of $t \cdot a$ – $i^\circ \cdot i = 1$ and $t \cdot i^\circ = t \cdot a$ – then $t \cdot i \cdot i^\circ = t \cdot a$, and hence, by the previous remark, $i \cdot i^\circ = a$. Conversely, let $r : X \to I$ be an arrow and suppose coreflexives split. Let $i \cdot i^\circ = \mathscr{D}(r)$, $i^\circ \cdot i = 1$ be a splitting of $\mathscr{D}(r)$; then $t \cdot i^\circ = t \cdot i \cdot i^\circ = t \cdot \mathscr{D}(r) = r$, again by the previous remark. Thus i tabulates r. □

A virtue of the notion of Cartesian bicategory is that it is stable under the splitting of idempotents, whereas Freyd's notion of allegory is only stable under the splitting of symmetric idempotents. The usual construction of the splitting of idempotents applied to **Rel**($\mathscr{E}$) simply as a category gives the bicategory of ordered objects in $\mathscr{E}$ and ideals between them. The following lemma will be used in the theory of ordered objects, as well as in discussing the free functional completion of a bicategory of relations.

Lemma 4.3. *Let* $\mathbb{B}$ *be a Cartesian bicategory and let* $\mathscr{I}$ *be the class of monads* (*comonads*) *in* $\mathbb{B}$. *Then*

(i) *the splitting* $\hat{\mathscr{I}}$ *of the monads* (*comonads*) *in* $\mathscr{I}$ *is a Cartesian bicategory* (*assuming* $\mathbb{B}$ *to be a 'bicategory of relations' in the comonad case*),

(ii) *if* $\mathbb{B}$ *is a 'bicategory of relations' and* $\mathscr{J}$ *is the class of symmetric monads* (*comonads*) *in* $\mathbb{B}$, *then* $\hat{\mathscr{J}}$ *is a 'bicategory of relations'.*

Proof. (i) Recall that $\hat{\mathscr{I}}$ has as objects the monads (comonads) of $\mathbb{B}$ and as arrows $r : p \to q$ the arrows r of $\mathbb{B}$ such that $rp = r = q \cdot r$. It is easy to see that $\hat{\mathscr{I}}$ is a bicategory (the identity arrows are $p : p \to p$). The tensor product in $\mathbb{B}$ induces one in $\hat{\mathscr{I}}$. In the monad case define $\Delta_p : p \to p \otimes p$ by $\Delta_p = (p \otimes p) \cdot \Delta$ and the adjoint Δ_p^* as $p \cdot \Delta^*$, t_p simply as t and the adjoint as t^*. In the comonad case define Δ_p as $\Delta \cdot p$ and the adjoint Δ_p^* as $p \cdot \Delta^\circ$, t_p as $t \cdot p$ and the adjoint $t^* \cdot p$ as $p \cdot t^\circ$. A straightforward calculation shows $\hat{\mathscr{I}}$ to be Cartesian.

(ii) We need only to prove axiom (D). First observe that if $p \subset i$, then p is a partial map, so that $\Delta \cdot p = (p \otimes p) \cdot \Delta$. Hence in each case $\Delta_p \cdot \Delta_p^* = (p \otimes p) \cdot \Delta \cdot \Delta^\circ \cdot (p \otimes p)$. Now, in the comonad case, this last is $(p \otimes p\Delta^\circ) \cdot (p \otimes \Delta \cdot p)$, that is $(1_p \otimes \Delta_p^*) \cdot (\Delta_p \otimes 1_p)$. In the monad case, first observe that from the basic law of Theorem 2.4, we have $(p \otimes p) \cdot \Delta = (1 \otimes p)\Delta_p$. Then

$$\begin{aligned}
\Delta_p \cdot \Delta_p^* &= (1 \otimes p)(1 \otimes \Delta^\circ) \cdot (p \otimes 1) \cdot (1 \otimes p) \cdot (\Delta \otimes 1) \cdot (p \otimes 1) \\
&= (1 \otimes p) \cdot (1 \otimes \Delta^\circ) \cdot (p \otimes 1 \otimes p) \cdot (1 \otimes \Delta) \cdot (p \otimes 1) \\
&= (p \otimes (p \cdot \Delta^\circ \cdot (1 \otimes p)) \cdot ((p \otimes 1) \cdot \Delta \cdot p) \otimes p) \\
&= (1_p \otimes \Delta_p^*) \cdot (\Delta_p \otimes 1_p). \qquad \square
\end{aligned}$$

Remarks 4.4. (i) If $\mathbb{B}$ is a 'bicategory of relations', then taking $\mathscr{I}$ to be the class of coreflexives Cor($\mathbb{B}$) of $\mathbb{B}$ the splitting Cor($\mathbb{B}$)$\hat{}$ is the free functional completion of $\mathbb{B}$: $\mathscr{E} =$ **Map**(Cor($\mathbb{B}$)$\hat{}$) is a regular category, and there is a natural equivalence between the category of monoidal homomorphisms $\mathbb{B}\to\mathbf{Rel}(S)$, S a regular category, and the category of left exact image-preserving functors $\mathscr{E}\to S$. In particular, if $\mathbb{B}$ is the 'bicategory of relations' associated to a theory $\mathbb{T}$, then $\mathscr{E}=\mathbf{Map}(\mathrm{Cor}(\mathbb{B})\hat{\ })$ is the logical category $\mathscr{E}(\mathbb{T})$ syntactically constructed in [13] or [9].

(ii) If $\mathbb{B}=\mathbf{Rel}(\mathscr{E})$, define $\mathbf{Mon}(\mathbb{B})$ $(=\mathbf{Ord}(\mathscr{E}))$ as the splitting of all monads in $\mathbb{B}$. Clearly all monads in $\mathbf{Mon}(\mathbb{B})$ have a splitting (that is a Kleisli construction) and $\mathbf{Mon}(\mathbb{B})$ is the free completion of $\mathbb{B}$ with respect to the Kleisli construction. An easy computation shows that a bicategory $\mathbb{B}$ has the Kleisli construction for monads in $\mathbb{B}$ iff the canonical embedding $\mathbb{B}\to\mathbf{Mon}(\mathbb{B})$ is an equivalence. (For the details of a more general result see [5].) Hence the construction $\mathbf{Mon}(-)$ is idempotent. By the previous theorem $\mathbf{Mon}(\mathbb{B})$ is Cartesian, if $\mathbb{B}$ is. In fact more is true:

Lemma 4.5. *If $\mathbb{B}$ is a 'bicategory of relations', then* $\mathbf{Mon}(\mathbb{B})$ *is compact closed, and canonically* $((-)^\circ)^\circ=1$, $(-)^\circ\otimes(-)^\circ=(-\otimes-)^\circ$ (*actual equality*!).

Proof. Define $(X,p)^\circ=(X,p^\circ)$ and

$$\eta_{(X,p)}=t\to(X,p)^\circ\otimes(X,p)=(p^\circ\otimes p)\Delta t^\circ=(p^\circ\otimes p)\cdot\eta,$$

$$\varepsilon_{(X,p)}=(X,p)\otimes(X,p)^\circ\to I=t\cdot\Delta^\circ(p\otimes p^\circ)=\varepsilon_X\cdot(p\otimes p^\circ).$$

A straightforward computation using the basic law of Theorem 2.4 shows that $(X,p)\dashv(X,p)^\circ$. $\square$

If $\mathbb{B}$ is a 'bicategory of relations', define $\mathbf{Eq}(\mathbb{B})$ as the splitting of symmetric monads in $\mathbb{B}$. Then $\mathbf{Eq}(\mathbb{B})$ is again a 'bicategory of relations'. If $\mathbb{B}=\mathbf{Rel}(\mathscr{E})$, then $\mathbf{Eq}(\mathbb{B})$ is the bicategory of equivalence relations in $\mathscr{E}$ and compatible relations between them. Recalling that a regular category is *exact* if every equivalence relation in $\mathscr{E}$ has a coequalizer e whose kernel is the given equivalence relation, it can easily be seen that $\mathscr{E}$ is exact just when in $\mathbb{B}=\mathbf{Rel}(\mathscr{E})$ every symmetric monad has a Kleisli construction; that is, iff symmetric monads split in $\mathbb{B}$. Following Freyd, we will call such bicategories *effective*. From the above discussion the following characterization theorem clearly emerges:

Theorem 4.6. *A bicategory $\mathbb{B}$ is biequivalent to a bicategory* $\mathbf{Ord}(\mathscr{E})$ *of ordered objects (and ideals between them) in an exact category $\mathscr{E}$ iff*

(i) *$\mathbb{B}$ is Cartesian*;

(ii) *every monad in $\mathbb{B}$ has a Kleisli construction*;

(iii) *for each object X in $\mathbb{B}$ there exists a discrete object X_0 and a monad $X\to X$ whose Kleisli construction is isomorphic to X_0*;

(iv) *if $a\subseteq 1_X$ and X is discrete, then a splits.*

Proof. If $\mathbb{B}$ is such a bicategory, then define $\mathbb{B}_0$ as the bicategory of discrete objects. Then $\mathbb{B}_0$ is functionally complete. To see this we need to show that if $a\subseteq 1_X$,

X discrete, and if $i^*i=a$, $i\cdot i^*=1$ is a splitting of a, then the domain X' of $i:X'\to X$ is discrete. But

$$\begin{aligned}\Delta_{X'}\cdot\Delta^*_{X'}&=\Delta_{X'}\cdot i^*\cdot i\Delta^*_{X'}=(i^*\otimes i^*)\cdot\Delta_X\cdot\Delta^*_X(i\otimes i)\\&=(i^*\otimes i^*)(1\otimes\Delta^*)\cdot(\Delta\otimes 1)\cdot(i\otimes i)=(i^*\otimes i^*\Delta^*)\cdot(\Delta i\otimes i)\\&=(i^*\otimes(i^*\otimes\Delta^*\cdot i^*))\cdot((i\otimes i)\otimes\Delta\cdot i)\\&=(1\otimes\Delta^*)\cdot(i^*\otimes i^*\otimes i^*)\cdot(i\otimes i\otimes i)(\Delta\otimes 1)\\&=(1\otimes\Delta^*)\cdot(i^*i\otimes i^*i\otimes i^*i)\cdot(\Delta\otimes 1)=(1\otimes\Delta^*)(\Delta\otimes 1).\end{aligned}$$

A similar argument shows that if $p:X\to X$ is an equivalence relation in $\mathbb{B}_0$, then the splitting $X\to X_p$ of p stated in (ii) still has discrete codomain. Thus $\mathscr{E}=\mathbf{Map}(\mathbb{B}_0)$ is an exact category, and conditions (ii), (iii) ensure that $\mathbf{Ord}(\mathscr{E})$ is equivalent to $\mathbb{B}$: $\mathbf{Mon}(\mathbb{B}_0)$ is a full subbicategory of $\mathbf{Mon}(\mathbb{B})$; from (ii) and Remark 4.4(ii), $\mathbb{B}$ is biequivalent to $\mathbf{Mon}(\mathbb{B})$; from (iii), the resulting homomorphism $\mathbf{Mon}(\mathbb{B}_0)\to\mathbb{B}$ is a biequivalence. □

Remark 4.7. If $\mathscr{E}$ is just a regular category, then $\mathbf{Mon}(\mathbf{Rel}(\mathscr{E}))$ can be constructed and satisfies (i)–(iv) of Theorem 4.6. However, the discrete objects in $\mathbf{Mon}(\mathbf{Rel}(\mathscr{E}))$ can be shown to be equivalence relations in $\mathscr{E}$, that is, the objects of the free exact category over the regular category $\mathscr{E}$. This explains why in the characterization theorem we assume $\mathscr{E}$ exact.

5. Abelian bicategories

Additive relations – relations in abelian categories – have been studied by various authors, but the only characterization known to the present authors is the early one of Puppe [14] which predates the notion of exact category and is thus rather complicated. The notion of 'bicategory of relations' leads to a simple and perfectly self-dual characterization of such bicategories.

Definition 5.1. Let $\mathbb{B}$ be a bicategory with tensor product. $\mathbb{B}$ is an *abelian bicategory* if both $\mathbb{B}$ and $\mathbb{B}^{co}$ (2-cells reversed) are 'bicategories of relations' with respect to the (same) given tensor product.

An abelian bicategory is functionally complete if both $\mathbb{B}$ and $\mathbb{B}^{co}$ are functionally complete as 'bicategories of relations'.

We will denote by $O_X:I\to X$ and $\delta_X:X\otimes X\to X$ the maps which provide the Cartesian structure on $\mathbb{B}^{co}$.

Theorem 5.2. *Bicategories of relations of abelian categories are characterized by the property that they are functionally complete abelian bicategories.*

Proof. To prove that $\mathbb{B}=\mathbf{Rel}(\mathscr{E})$, $\mathscr{E}$ an abelian category, is a functionally complete abelian bicategory let us first observe that $\mathbb{B}^{co}$ being a bicategory of relations means that each object in $\mathbb{B}$ is equipped with a cocommutative comonoid structure such that the comultiplication and the counit are right adjoints and are thus opposites of maps $X\otimes X\to X$ and $I\to X$. The opposite should be understood as the involution arising from the compact closed structure on $\mathbb{B}$. These maps are a commutative monoid structure on X. The codiagonal and the fact that the terminal is also initial provide such a structure on each object of $\mathbb{B}=\mathbf{Rel}(\mathscr{E})$. Condition (U) $(\delta\cdot(r\otimes r)\subset r\cdot\delta)$ is easily checked. Discreteness of X in $\mathbb{B}^{co}$ $(\delta^\circ\cdot\delta=(1\otimes\delta)\cdot(\delta^\circ\otimes 1))$ follows from the fact that the square

$$\begin{array}{ccc} X\otimes X\otimes X & \xrightarrow{\delta\otimes 1} & X\otimes X \\ {\scriptstyle 1\otimes\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ X\otimes X & \xrightarrow{\delta} & X \end{array}$$

is a pullback in $\mathscr{E}$ (additivity).

For $\mathbb{B}^{co}$ to be functionally complete means that if $r\supseteq 1$, then r splits. But this follows from the well-known fact that in an abelian category $\mathscr{E}$ every reflexive relation is an equivalence relation [1]. Since $\mathscr{E}$ is exact, the coequalizer e of r gives a splitting of r in $\mathbb{B}$ $=\mathbf{Rel}(\mathscr{E})$.

Conversely, if $\mathbb{B}$ is a functionally complete abelian bicategory, then $\mathscr{E}=\mathbf{Map}(\mathbb{B})$ is certainly a regular category and the structure on $\mathbb{B}^{co}$ provides the semiadditivity (coproducts = products) of $\mathscr{E}$ by the dual of Theorem 1.6. The discreteness of $\mathbb{B}^{co}$ just says that the arrows

$$\bar{\eta}_X=I\xrightarrow{O_X}X\xrightarrow{\delta_X^\circ}X\otimes X,$$

$$\bar{\varepsilon}_X=X\otimes X\xrightarrow{\delta_X}X\xrightarrow{O_X^\circ}I$$

give rise to a compact closed structure on $\mathbb{B}$, by the dual of Theorem 2.4. Since the two compact closed structures are naturally equivalent by standard arguments on adjunctions, there exists a unique isomorphism $V_X:X\to X$ such that $(1\otimes V_X)\cdot\bar{\eta}_X=\eta_X$ and $\bar{\varepsilon}_X\cdot(1\otimes V_X)=\varepsilon_X$. The condition $(1\otimes V_X)\cdot\bar{\eta}_X=\eta_X$ qualifies V_X as the map $X\to X$ which gives the group structure on each object X, since it means that

$$\begin{array}{ccccc} X & & \xrightarrow{\quad t_X\quad} & & I \\ {\scriptstyle\Delta}\downarrow & & & & \downarrow{\scriptstyle O_X} \\ X\otimes X & \xrightarrow{1\otimes V} & X\otimes X & \xrightarrow{\delta} & X \end{array}$$

is a pullback. Thus $\langle X,\delta,O_X,V_X\rangle$ is an abelian group object in $\mathbf{Map}(\mathbb{B})$ and

$V^2=1$. Thus $\mathscr{E}=\mathbf{Map}(\mathbb{B})$ is additive, and it is also exact by the functional completeness of $\mathbb{B}^{co}$. By Tierney's theorem that an exact additive category is abelian, $\mathscr{E}$ is an abelian category. □

Remarks 5.3. (i) By the dual of Theorem 1.6, the Cartesian structure of $\mathbb{B}^{co}$ provides a local union on $\mathbb{B}$ defined as $r\cup s=\delta\cdot(r\otimes s)\cdot\delta^{\circ}$, for which the zero map $X\xrightarrow{O_X}I\xrightarrow{O_Y^{\circ}}Y$ is a minimal element. Moreover, the compact closed structure on $\mathbb{B}$ induced from the structure on $\mathbb{B}^{\circ}$ yields an involution which coincides with the one given by the Cartesian structure on $\mathbb{B}$. (Just use the very definition of the involution and the fact that $\varepsilon\cdot(1\otimes V)=\varepsilon(V\otimes 1)$, since $t^{\circ}\cdot\Delta(1\otimes V)=t^{\circ}\cdot V\cdot\Delta(V^{\circ}\otimes 1)=t^{\circ}\cdot\Delta\cdot(V\otimes 1)$, because $V^2=1$ implies $V^{-1}=V$; but $V^{-1}=V$. Similarly $(1\otimes V)\eta=(V\otimes 1)\eta$.) So, by the dual of Theorem 2.4 the bicategory $\mathbb{B}$ satisfies the dual of Freyd's modular law $(s\cdot r\cup t\supseteq(s\cup t\cdot r^{\circ})\cdot r)$ as well as the duals of Lemma 2.5, Corollary 2.6, 2.7, ... In particular, it follows from the dual of the modular law that the lattice of subobjects of each object of $\mathbf{Map}(\mathbb{B})$ is modular.

(ii) The most we can say about the interplay between composition and local unions in an abelian bicategory is the dual of the modular law. It is always true that if f is a map, then $(r\cup s)\cdot f^{\circ}=r\cdot f^{\circ}\cup s\cdot f^{\circ}$ and $f\cdot(r\cup s)=fr\cup fs$. Logically this means that $\exists$ commutes with $\cup$. But it is not true that $(r\cup s)\cdot f=rf\cup sf$ or $f^{\circ}\cdot(r\cup s)=f^{\circ}\cdot r\cup f^{\circ}\cdot s$, as well as $0\cdot f^{\circ}=0$; these conditions mean that substition preserves finite unions. Any of these conditions would imply that $\mathbb{B}$ is degenerate (Adelman).

The category $\mathscr{H}$ of Hilbert spaces and continous linear maps is regular and additive, so that $\mathbb{B}=\mathbf{Rel}(\mathscr{H})$ is an abelian bicategory where just functional completeness of $\mathbb{B}^{co}$ fails.

(iii) Corollary 3.6 extends also to the abelian case. If $\mathscr{E}$ and $\mathscr{F}$ are abelian categories, then the category of additive exact functors $\mathscr{E}\to\mathscr{F}$ is equivalent to the category of monoidal homomorphisms of bicategories $\mathbf{Rel}(\mathscr{E})\to\mathbf{Rel}(\mathscr{F})$.

6. Matrices

In this section we will characterize bicategories of relations of Grothendieck toposes.

In studying bicategories of relations of regular categories we have seen that products in the category lift to tensor products in the bicategory of relations. But, once the bicategory of relations can be constructed and 'good' sums exist in the category, then sums lift to the bicategory of relations: if $\mathbb{B}$ is a regular category with (finite) disjoint and pullback-stable sums, then $\mathbf{Rel}(\mathbb{B})$ has (finite) bicoproducts and, conversely

Theorem 6.1. *If $\mathbb{B}$ is a functionally complete 'bicategory of relations' having (finite) bicoproducts, then $\mathscr{E}=\mathbf{Map}(\mathbb{B})$ has (finite) disjoint and pullback-stable coproducts.*

Proof. For simplicity we give the proof for the finite case. To show that $\mathscr{E} = \mathbf{Map}(\mathbb{B})$ has coproducts we need to prove that the initial arrow $O_X : O \to X$ and the codiagonal $\delta_X : X \oplus X \to X$ are maps (O and $\oplus$ denote initial and sum in $\mathbb{B}$). By using Lemma 2.5 it is enough to prove that the two arrows are comonoid homomorphisms: the needed equations for O_X follow because O is initial and $O_I = t_O$; the ones for δ_X follow from the fact that the injections i are now maps such that $\delta \cdot i = 1$. So, to prove $\Delta_X \cdot \delta_X = (\delta_X \otimes \delta_X)\Delta_{X \oplus X}$, it is enough to prove that the two arrows are the same when composed with injections, that is, that $\Delta_X \cdot \delta_X \cdot i = \Delta_X$, and $(\delta_X \otimes \delta_X) \cdot \Delta_{X \oplus X} \cdot i =$ (because i is a map) $= (\delta_X \cdot i \otimes \delta_X \cdot i) \cdot \Delta_X = \Delta$. Finally, the counit preservation follows from the fact that $t_{X \oplus X} = t_{I \oplus I} \cdot (t_X \oplus t_X)$ and $t_{I \oplus I} = \delta_I$. Thus $\mathscr{E} = \mathbf{Map}(\mathbb{B})$ has coproducts, and with the same argument as in Theorem 1.6 we can show that the definitions

$$r \cup s = \delta \cdot (r \oplus s)\delta^\circ, \qquad O_{X,Y} = O_Y \cdot O_X^\circ$$

give local unions and initials which, moreover, are stable on both sides because of the involution on $\mathbb{B}$.

Observe that the compact closedness of $\mathbb{B}$, and the fact that the involution is the identity on objects, imply that bicoproducts in $\mathbb{B}$ are also biproducts. With standard computations based on the previous facts it can be shown that initial maps $O_X : O \to X$ are monos ($O_X^\circ \cdot O_X = 1$). Thus injections i_X are also mono ($i_X^\circ \cdot i_X = 1$) as well as jointly epic ($i_X \cdot i_X^\circ \cup i_Y \cdot i_Y^\circ = 1$) and disjoint ($i_Y^\circ \cdot i_X = O_{X,Y}$). As for the stability of sums under pullbacks in $\mathscr{E} = \mathbf{Map}(\mathbb{B})$, first observe that $X \otimes -$, having a right adjoint in $\mathbb{B}$, preserves all colimits which exist in $\mathbb{B}$, thus sums in $\mathbb{B}$, hence also in $\mathscr{E}$. We just need to show that the lattice of subobjects of an object in $\mathscr{E}$ is distributive. Since this lattice is isomorphic to the lattice of coreflexives on the object in $\mathbb{B}$ (see Lemma 4.2) and composition of coreflexives reduces to intersections (see Lemma 4.1), this last reduces to the stability of local unions under composition. □

Remark 6.2. In proving Theorem 6.1, we showed that the assumption that $\mathbb{B}$ has bicoproducts and the compact closedness of $\mathbb{B}$, force $\mathbb{B}$ to be 'semiadditive' – initial $O \to X$ and codiagonal $X \oplus X \to X$ arrows are maps such that the adjoints provide bicoproducts with a structure of biproducts. Hence $\mathbb{B}$ is a 'distributive bicategory' (see Remark 3.7). Conversely, given such a bicategory $\mathbb{B}$ we can construct the *free semiadditive bicategory* $\mathbf{Matr}(\mathbb{B})$ as follows:
- *objects* are families $e : X \to |\mathbb{B}|$ of objects of $\mathbb{B}$;
- *arrows* $r : (X, e) \to (Y, e')$ are matrices $r(x, y) : e(x) \to e(y)$ of arrows of $\mathbb{B}$;
- *composition of arrows* is matrix composition;
- 2-*cells* are defined pointwise.

$\mathbf{Matr}(\mathbb{B})$ enjoys the following remarkable properties:
(a) $\mathbf{Matr}(\mathbb{B})$ is semiadditive;
(b) if $\mathscr{W}$ is semiadditive, then $\mathscr{W} \xrightarrow{\sim} \mathbf{Matr}(\mathscr{W})$ and thus the construction of

Matr$(-)$ is idempotent (see [5]);

(c) if $\mathbb{B}$ is Cartesian, then **Matr**($\mathbb{B}$) is Cartesian (define $(X,e)\otimes(Y,e')$ as $(X\times Y, e\otimes e')$, where $(e\otimes e')(x,y)=e(x)\otimes e'(y)$, and similarly for arrows);

(d) if $\mathbb{B}$ is a 'bicategory of relations', then so also is **Matr**($\mathbb{B}$).

As a consequence of property (d), if $\mathscr{E}$ is a geometric category, then **Map(Matr(Rel(**$\mathscr{E}$**)))** is the free geometric category having all small indexed disjoint and universal sums. By applying Property (d) and Theorem 6.1, we just need to check that if $\mathbb{B}$ is functionally complete, then **Matr**($\mathbb{B}$) is such. For details of this and related results see [3].

Theorem 6.1 applies to bicategories of relations of a Grothendieck topos as follows:

Theorem 6.3. *A bicategory* $\mathbb{B}$ *is of the form* **Rel**($\mathscr{E}$) *with* $\mathscr{E}$ *a Grothendieck topos iff*

(i) *it is a functionally complete 'bicategory of relations'*;

(ii) *it is effective*;

(iii) *it has small bicoproducts*;

(iv) *it has a small set* G *of generators* ($r\subset s: X\to Y$ *iff for all* $x: U\to X$, $U\in G$, $r\cdot x\subseteq s\cdot x$). $\square$

The proof relies on the previous theory of relations and the theorem of Giraud.

Remarks 6.4. (i) Observe that, starting with Giraud's axioms for a Grothendieck topos $\mathscr{E}$, the distributivity of **Rel**($\mathscr{E}$) gives immediately that $\mathscr{E}$ has all coequalizers: if $f,g: X\to Y$ is a parallel pair of maps, then the splitting of the free equivalence relation $r'=\bigcup_n r^n$ generated by $r=g\cdot f^\circ\cup f\cdot g^\circ$ is the coequalizer of f,g.

(ii) Another consequence of property (b), Corollary 3.6 and the above remark is the following: if $\mathscr{E},\mathscr{F}$ are Grothendieck toposes, then the category **Top**($\mathscr{E},\mathscr{F}$) is equivalent to the category of monoidal and local union preserving homomorphisms **Rel**($\mathscr{F}$)$\to$**Rel**($\mathscr{E}$) which is equivalent to *the category of sum and tensor preserving homomorphisms* **Rel**($\mathscr{F}$)$\to$**Rel**($\mathscr{E}$).

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